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Progress in Nonlinear Differential Equations and Their Applications

Gabriella Tarantello

Selfdual Gauge Field Vortices

An Analytical Approach

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Preface

Gauge Field theories (cf. [Ry], [Po], [Q], [Ru], [Fr], [ChNe], [Pol], and [AH]) have had a great impact in modern theoretical physics, as they keep internal symmetries and can account for important physical phenomena such as: spontaneous symmetry breaking (see e.g., [En1], [En2], and [Br]), the quantum Hall effect (see e.g., [GP], [Gi], [McD], [Fro], and [Sto]), charge fractionalization, superconductivity, and supergravity (see e.g., [Wi], [Le], [GL], [Park], [Sch], [Ti], and [KeS]). In these notes, we focus on specific examples of gauge field theories which admit a selfdual structure when the physical parameters satisfy a "critical" coupling condition that typically identifies a transition between different regimes. The selfdual regime is characterized by the presence of "special" soliton-type solutions corresponding to minimizers of the energy within certain constraints of "topological" nature. Such solutions, known as selfdual solutions, satisfy a set of first-order (selfdual) equations that furnish a "factorization" for the second-order gauge field equations. Furthermore, each class of "topologically equivalent" selfdual solutions form the space of moduli, whose characterization is one of the main objectives in gauge theory. The situation is nicely illustrated by the (classical) Yang–Mills gauge field theory (cf. [YM]), whose field equations (over S^4) are satisfied by selfdual/antiselfdual connections, simply by virtue of Bianchi identity.

The selfdual/antiselfdual connections define the well-known *instanton* solutions of the Yang–Mills field equations. They can be characterized by the property that, among all connections with prescribed second Chern number $N \in \mathbb{Z}$, the N-instantons identify those with minimum Yang–Mills energy. In this way one sees that every second Chern–Pontryagin class of S^4 can be represented by a family of instantons, which forms the space of moduli in this case. The space of moduli of N-instantons has been completely characterized in terms of the assigned second Chern number, N. In this respect, we refer to [AHS1], [Schw], and [JR] for a dimensional analysis of such space, and to [BPST], [ADHM], [AHS2], [JNR], [JR], ['tH1], and [Wit1] for explicit constructions of N-instantons. Although instantons do not exhaust the whole family of finite action solutions of the Yang–Mills equations (see e.g., [SSU], [Par], [SS], [Bor], [Bu1] [Bu2], and [Ta3]), their impact both in mathematics and in physics has been remarkable. From the mathematical point of view, it is enough to mention their striking implications toward the study of differential topology for four-dimensional manifolds (cf. [DK] and [FU]). In the selfdual context, we see through the work contained in [Fa], [FM2], [JT], [Le], [Ra], [RS], [AH], [Pe], [Wei], [NO], [Bra1], [Bra2], [GO], [PS], [Hi], and [Y1], how influential has been the study of Yang–Mills instantons toward the understanding of other selfdual soliton-type configurations including: monopoles, vortices, kinks, strings, etc.

In this respect, we shall be concerned with the case where, in account of the Higgs mechanism, we include a Higgs field in the theory, to be (weakly) coupled with the other gauge fields. In this way we are lead to consider the Yang–Mills–Higgs theory, where we still may attain selfduality by a dimensional reduction procedure yielding to *monopoles* as the three-dimensional soliton-solutions for the corresponding selfdual Yang–Mills–Higgs equations. We refer to [JT], [Le], [AH], [Pe], [GO], and [Y1] for a detailed discussion of monopoles based on their strong ties with instantons.

Another instructive example about selfduality is offered by the abelian Higgs model as pointed out by Bogomolnyi in [Bo]. More precisely, by considering a planar (abelian) Maxwell–Higgs theory with a scalar "double-well" potential (having appropriate strength), Bogomonlyi in [Bo] derives a set of selfdual equations, whose solutions describe the well-known *selfdual Maxwell–Higgs vortices* discussed in [NO]. One may regard these configurations as the cross-section of the "vortex-tubes" observed experimentally in superconductors which are subjected to an external magnetic field. In fact, the selfdual situation identified by Bogomonlyi describes the relativistic analog of the Ginzburg–Landau model in superconductivity (cf. [GL]), with parameters that correspond to the borderline case between Type I and Type II superconductors. In analogy with Yang–Mills instantons, it is possible to distinguish abelian Maxwell–Higgs selfdual vortices into distinct "topological" classes, relative to each element of the homotopy group of S^1. The role of S^1 in this context is readily explained, since topologically, it represents the abelian gauge group $U(1)$. More precisely, for every $N \in \pi_1(S^1) = \mathbb{Z}$, the family of selfdual Maxwell–Higgs N-vortices corresponds to the minima of the Maxwell–Higgs energy, as constrained to Higgs fields with topological degree N. Moreover, the space of moduli formed by N-vortices over a surface M has been completely characterized as a manifold equivalent to M^N modulo the group of permutations of N elements (see [Ta1], [JT], [Bra1] [Bra2], [Ga1], [Ga2], [Ga3], [WY] and [NO], [MNR], [HJS], [Y1] for further results). We mention that, in contrast to Yang–Mills instantons, selfdual Maxwell–Higgs vortices fully describe finite energy static solutions of the Maxwell–Higgs field equations. In this way, one deduces a complete characterization of Ginzburg–Landau vortices in the selfdual regime. More recently, much progress has also been made in the understanding of Ginzburg–Landau vortices away from the selfdual regime, as one may see for example in [BBH], [DGP], [Riv], and [PR].

In the early 1990s, Chern–Simons theories were introduced to account for new phenomena in condensed matter physics, anyon physics, supercondictivity, and supergravity (see [D1], [D3] and references therein). Selfduality entered this new scenario with a primary role. In fact, it became immediately clear that although the Chern–Simons term was extremely advantageous from the point of view of the physical applications, it introduced serious analytical difficulties that prevented the corresponding Chern–Simons gauge field equations from being handled with mathematical rigor. Thus, to gain a mathematical grasp of Chern–Simons-Higgs configurations, it has been useful to consider convenient selfdual first-order reductions of the (difficult) second-order Chern–Simons field equations. In this respect, following Bogomolnyi's approach, Jackiw–Weinberg [JW] and Hong–Kim–Pac [HKP] introduced an abelian Chern–Simons-Higgs 6th-order model that obeys a selfdual regime. For such a model, the Maxwell electrodynamics is replaced by the Chern–Simons electrodynamics and the Higgs "double-well" potential is replaced by a "triple-well" potential. We shall see how those characteristics give rise to a Chern–Simons theory which supports a rich class of selfdual vortices yet to be completely classified. Subsequently, other interesting selfdual Chern–Simons models were introduced both in relativistic and non-relativistic contexts and address the abelian and non-abelian situation. We refer to the excellent presentation on this subject, as provided by Dunne in [D1] and [D3].

The purpose of these notes is to illustrate the new and delicate analytical problems posed by the study of selfdual Chern–Simons vortices. We shall present the resolution of some vortex problems and discuss the many remaining open questions. By this analysis, we shall also become capable of handling the celebrated electroweak theory in relation to the selfdual regime characterized by Ambjorn–Olesen in [AO1], [AO2], and [AO3].

In Chapter 1, we introduce the basic mathematical language of gauge theory in order to formulate examples of Chern–Simons-Higgs theories both in the abelian and the non-abelian setting. We are going to compare their features with the well-known abelian Higgs and Yang–Mills–Higgs model. For those theories, we will see how to attain selfduality and to derive the relative selfdual equations that will represent the main objective of our study. In this perspective, we shall investigate also the electroweak theory of Glashow–Salam–Weinberg (see [La]) according to the selfdual ansatz introduced by Ambjorn–Olesen [AO1], [AO2], and [AO3]. In addition, we shall analyze selfgravitating electroweak strings, as they occur when we take into account the effect of gravity through the coupling of the electroweak field equations with Einstein equations.

In Chapter 2, we shall adopt the approach introduced by Taubes for the abelian Higgs model (see [Ta1] and [JT]) to reduce the selfdual field equations into elliptic problems involving exponential nonlinearities. Naturally, this will lead us to examine Liouville-type equations, whose solutions (see [Lio]) have entered already in the explicit construction of some special selfdual configurations, for example: Wittens' instantons (cf. [Wit1]), spherically symmetric monopoles (cf. [JT]) and Olesen's periodic one-vortices, (cf. [Ol]).

Unfortunately, our problems will not be explicitly solvable, and we shall need to introduce sophisticated analytical tools in order to obtain solutions whose features can be described consistently with the physical applications. To this end, we shall recall some known facts about Liuoville-type equations, in relation to their mean field formulations and to the Moser–Trudinger inequality (cf. [Au]). This material is collected in Chapter 2 together with a general discussion on related mathematical problems and applications.

On the basis of this information, we proceed in Chapter 3 to analyze planar Chern–Simons vortices for the abelian 6th-order model proposed by Jackiw–Weinberg [JW] and Hong–Kim–Pac [HKP]. Here, we encounter the first novel feature of Chern–Simons vortices in comparison with Maxwell–Higgs vortices. In fact, we are now dealing with a theory that admits both symmetric and asymmetric vacua states, and hence we expect multiple vortex configurations to occur according to the nature of the vacuum by which they are supported. More precisely, in the planar case this amounts to classify vortices in relation to their asymptotic behavior at infinity. Thus, we shall call "topological" those vortices that at infinity are asymptotically gauge equivalent to an asymmetric vacuum state; while "non-topological" will be called those that at infinity are asymptotically gauge equivalent to the symmetric vacuum. This terminology is justified by the fact that only the topological solutions correspond to minimizers of the Chern–Simons-Higgs energy within the class of Higgs fields with assigned topological degree. Observe that since the Maxwell–Higgs model admits only asymmetric vacua states, it can only support topological vortices. The goal of Chapter 3 is to show rigorously that indeed both "topological" and "non-topological" vortices actually coexist in the Chern–Simons-Higgs theory proposed in [JW] and [HKP]. In particular, in Section 3.2 we show that the topological ones are very much equivalent to the Maxwell–Higgs vortices, with whom they share the same uniqueness property and asymptotic behavior at infinity. It is interesting to note however that the "topological" Chern–Simons vortices are by no means "approximations" of the Maxwell–Higgs vortices (in any reasonable sense), as one finds out from some limiting properties (see property (3.1.3.) (c)). In section 3.3 we shall present the construction of Chae–Imanuvilov [ChI1] relative to non-topological Chern–Simons vortices which extends and completes that of Spruck–Yang [SY1] relative to the radially symmetric case. We also briefly describe the alternative construction of Chan–Fu–Lin [CFL] yielding to non-topological Chern–Simons planar vortices satisfying a nice "concentration" property around the vortex points (i.e., the zeroes of the Higgs field), consistently to what has been observed experimentally in the physical applications.

However, in contrast to the topological case, a complete asymptotic description of "non-topological" vortices is still under investigation. In fact, we are still far from a full characterization of selfdual Chern–Simons vortices in the same spirit of what is available for the Maxwell–Higgs model. For instance, we observe that our study does not clarify whether the Chern–Simons field equations admit (finite action) solutions other than the selfdual ones; nor does it even justify that this can never be the case when some symmetry assumption holds, as it occurs for instantons and monopoles.

The situation is even less clear for other models, for example, the abelian Maxwell-Chern–Simons-Higgs model proposed by Lee–Lee–Min [LLM] as a unified theory for both Maxwell–Higgs and Chern–Simons-Higgs models. For this model the asymptotic distinction between topological and non-topological vortices carries over, and the existence of such configurations has been established repectively in [ChK1] and [ChI3]. But a full classification of selfdual Maxwell-Chern–Simons-Higgs vortices is still missing, including the validity of a possible uniqueness property for the "topological" ones. See [ChN] for some contribution in this direction.

In the non-abelian framework, an interesting selfdual Chern–Simons model has been proposed by Dunne in [D2] (see also [D1]). Vortices in this case are expected to have an even richer structure, since now the system involves "intermediate" vacua states that interpolate between the symmetric and totally asymmetric ones. So far, it has been possible to establish only the existence of a planar non-abelian Chern–Simons vortex of topological nature (see [Y6]). The difficulty in the study of non-abelian vortices arises from the fact that they involve elliptic problems in a system form (see (2.1.21) and (2.1.25)) which introduces additional technical difficulties, as compared to the single Liouville-type equation arising from the Chern–Simons 6th-order model of [JW] and [HKP].

In Chapter 4, we still consider vortices for the Chern–Simons 6th-order model in [JW] and [HKP], but now we analyze them under (gauge invariant) periodic boundary conditions. This is motivated by the fact that lattice structures are likely to form in a condensed matter system. Also, they should account for the so-called "mixed states", that Abrikosov described to occur in superconductivity. He also anticipated their periodic structure long before they were observed experimentally (cf. [Ab]).

All periodic selfdual Chern–Simons vortices correspond to minima of the Chern–Simons energy in the class of Higgs fields with assigned topological degree. Therefore, as for the abelian Higgs model, for any given integer N, the moduli space of periodic Chern–Simons N-vortices is formed by minimizers of the energy among Higgs fields of fixed topological degree N. This space of moduli turns out to be much richer than that of the Maxwell–Higgs model described in [WY]. Indeed, we see that in the periodic case, a vortex must approach a vacuum state as the Chern–Simons coupling parameter tends to zero. Again, the presence of different vacua (asymmetric and symmetric) gives rise to asymptotically different vortex behaviors and this leads to multiplicity. Thus, in analogy to the planar case, we shall call of "topological-type" those vortices asymptotically gauge equivalent to asymmetric vacua states and of "non-topological-type" those vortices asymptotically gauge equivalent to the symmetric vacuum, when the Chern–Simons coupling parameter tends to zero. We shall introduce a useful criterion (see [DJLPW]) to distinguish between this different class of vortices and show, that indeed, both types coexist for the Chern–Simons 6th-order model in [JW] and [HKP].

The construction of "topological-type" vortices follows by using a minimization principle in the same spirit of that introduced for planar topological (Chern–Simons or Maxwell–Higgs) vortices. This approach allows us to clarify why a uniqueness property should hold for "topological-type" vortices, as recently established in [T7]. In fact, one sees that whenever a vortex is asymptotically gauge related to an asymmetric vacuum state, then from a variational point of view, it must correspond to a local minimum for the associated action functional (possibly for small values of the Chern–Simons coupling parameter). With this information in hand, it is then possible to check that uniqueness must hold (for details see [T7] and also [Cho], [ChN]).

Moreover, for our variational problem it is also possible to carry out a "mountain-pass" construction (cf. [AR]), and this permits us to deduce the existence of a "non-topological-type" vortex. Unfortunately, such construction guarantees convergence for the "non-topological-type" vortex toward the symmetric vacuum state only in

L^p-norm, and so it leaves out important information about the behavior of the vortex solution near the "vortex points." Thus, to improve such convergence to hold at least in uniform norm, we discuss an alternative construction (introduced in [T1] and [NT3]) that permits us to handle single or double vortices. Interestingly, in the double-vortex case such construction relates the Chern–Simons vortex problem to a mean field equation of the Liouville-type, which also enters into other contexts such as the study of extremals for the Moser–Trudinger inequality or the assigned Gauss curvature problem (see e.g., [ChY3] and references therein). As a matter of fact, our "non-topological-type" double-vortex solutions correspond to the "best" minimizing sequence for the Moser–Trudinger inequality on the flat torus. Thus, to control their behavior (as the Chern–Simons parameter tends to zero) it is necessary to develop a detailed blow-up analysis concerning solutions of Liouville-type equations in the presence of "singular" Dirac measures supported at the vortex points. This task will be the carried out in Chapter 5.

But before orienting our discussion towards those technical analytical aspects, we shall complete Chapter 4 by extending our approach to the study of periodic non-abelian Chern–Simons vortices. In this way, we land naturally on the field of elliptic systems of the Liouville-type, for which much more needs to be understood and clarified. For this reason, our contributions to the understanding of periodic non-abelian Chern–Simons vortices provide us only with partial answers and leave out much room for improvements. Mainly, we shall be concerned with the case of $SU(n+1)$-vortices, whose analysis involves the study of an elliptic Toda-lattice system characterized by many elements of analytical interest (see [JoW1], [JoW2] and [JoLW]).

In Chapter 5 we discuss the blow-up behavior of solution-sequences for Liouville-type equations in the presence of "singular" sources. The aim of this chapter is to present a systematic extension of the work of Brezis–Merle [BM], Li–Shafrir [LS] Brezis–Li–Shafrir [BLS] and Li [L2] concerning the "regular" case, namely, when singular sources are not taken into account. As it turns out, this task becomes rather delicate when blow-up occurs at the "singular" set, a situation likely to occur for our vortex solutions. In this case, the character of the "concentration" phenomenon is complicated by more degenerate aspects. Nonetheless, it is still possible to obtain sharp concentration/compactness principles, Harnack type inequalities, inf + sup estimates and "quantized" properties, which furnish a rather complete description of the blow-up phenomenon, as one expects to occur for the vortex solutions.

The material presented in this section is rather technical, and only concerns the case of a single equation. However, the presence of singular sources gives us the chance to introduce many technical tools which we hope may help in the blow-up analysis for systems (both in the "regular" and "singular" case), as well as, for related problems (cf. [Ci], [CP], [KS], [Mu], [CLMP1],[CLMP2], [Ki1], [Ki2], and [Wo]).

In Chapter 6, we take advantage of the analysis developed in Chapter 5 in order to complete the study of the asymptotic behavior of Chern–Simons periodic double-vortices as constructed in Chapter 4. As a by product of this analysis, we shall be able to obtain extremals for the Moser–Trudinger inequality on the flat two-torus.

Also, the information in Chapter 5, in combination with some variational techniques, will allow us to establish a general existence result for an elliptic system of interest in the study of selfdual electroweak periodic vortices.

The objective of Chapter 7 is to establish the existence of selfdual electroweak configurations of Abrikosov's "mixed-type" vortices and of self-gravitating electroweak strings. We will follow the path opened in the study of Chern–Simons vortices to obtain results in electroweak theory by means of the methods and techniques introduced in the previous chapters. There again, our analysis suffices to handle only a certain range of parameters, and it would be extremely useful to know whether or not our results extend to cover the full range of admissible parameters.

In this monograph, we have chosen to discuss only a few selected selfdual models which in our view, most effectively illustrate the advantage of the analytical approach that is pursued here and that originated in the work of Taubes ([Ta1] and [JT]). As already mentioned, this line of investigation has proved equally successful in the study of many other selfdual gauge field configurations. However, for the models considered here, the progress achieved in the selfdual case is particularly remarkable since rigorous mathematical results away from the selfdual regime remain rather scarce (see e.g., [HaK], [KS1], and [KS2]).

While we have provided indications on possible extensions of the given results to related models, we refer to the monograph of Yang [Y1] for a systematic use of Taubes' strategy to treat selfdual solutions arising in different physical contexts.

In fact, we hope that the reader can profit from the analysis developed here and further pursue the investigation of a variety of selfdual models, after the work in [Y1].

We have not touched upon other aspects related to selfduality, for example: "integrability" issues (see e.g., [Das], and [Hop]), dynamical properties of seldfual vortices (see e.g., [KL], [Ma], and [BL]), and solvability of initial value problems (see e.g., [Ch5], [ChC], and [ChCh2]). All of these problems pose very attractive and stimulating mathematical questions, and certainly deserve a lot more attention.

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Contents

1

Selfdual Gauge Field Theories

1.1 Introduction

In this chapter we introduce the reader to the gauge theory formalism in order to furnish examples of gauge field theories that support a selfdual structure.

We start with the simpler abelian situation, where most of the technical aspects of group representation theory can be avoided.

From the physical point of view, an abelian gauge field theory describes electromagnetic particle interactions. Thus we shall start by discussing the abelian Maxwell–Higgs model, well-known also as the relativistic counterpart of the Ginzburg–Landau model in superconductivity (cf. [GL]). We will illustrate Bogomolnyi's approach (cf. [Bo]) and attain selfduality for this model with parameters that describe the borderline case that distinguishes between type I and type II superconductors.

Next we will see how, in the same spirit, one can attain selfduality in the presence of the Chern–Simons term (cf. [D1]). In this context, we will focus on the "pure" Chern–Simons 6th-order model of Jackiw–Weinberg [JW] and Hong–Kim–Pac [HKP] and on the Maxwell–Chern–Simons–Higgs model of Lee–Lee–Min [LLM]. Subsequently, we will turn to the treatment of non-abelian gauge theories, and for this purpose we shall need to recall some basic facts about the representation of compact (semisimple) Lie groups (see [Ca], [Hu], and [Fe]). In the non-abelian framework, we shall formulate the Yang–Mills and the Yang–Mills–Higgs theories, as well as the non-abelian Chern–Simons theory in [D2] and the celebrated electroweak theory of Glashow–Salam–Weinberg (cf. [La]). We will show how to attain selfduality for such non-abelian models, which represent only a "sample" of the ample list of selfdual (relativistic) gauge field theories available in physics literature. Some extensions of the models considered here are contained in [KLL], [KiKi], [Kh], [CaL], [CG], [Wit2], [Va], and [D1]. In any case, we refer the interested reader to the monograph by Yang ([Y1]) for a broader discussion of relativistic and non-relativistic selfdual theories that model a wide range of physical phenomena.

1.2 The abelian Maxwell–Higgs and Chern–Simons theories

The abelian Maxwell–Higgs (or simply abelian-Higgs) and Chern–Simons theories describe electromagnetic interactions and, as gauge field theories, they are formulated by a Lagrangean density $\mathcal{L}$ expressed in terms of the *gauge potential* $\mathcal{A}$ and the Higgs (matter) field ϕ. Occasionally a neutral field is also included.

We focus our attention on euclidean theories formulated over the Minkowski space $(\mathbb{R}^{1+d}, g)$ with metric tensor $g =$ diag $(1, -1, \ldots, -1)$, where we denote by x^0 the time variable and $(x^1, \ldots, x^d)$ the space variables.

Usually, we shall use Greek indices to identify indifferently space or time variables, while the roman letters will be specific to space variables. We adopt standard notations and use the matrices $g = (g_{\alpha\beta})$ in the usual way to raise or lower indices, and let $g^{-1} = (g^{\alpha\beta})$ for the inverse. Moreover we use the summation convention over repeated lower and upper indices.

In this context, the *potential field* $\mathcal{A}$ is specified by its smooth real components

$$\mathcal{A}_\alpha : \mathbb{R}^{1+d} \to \mathbb{R}, \qquad \alpha = 0, 1, \ldots, d; \tag{1.2.1}$$

whereas, the *Higgs field* ϕ is a smooth complex valued function

$$\phi : \mathbb{R}^{1+d} \to \mathbb{C}. \tag{1.2.2}$$

To be consistent with the theoretical gauge-formalism (see next section), we identify the potential field $\mathcal{A}$ with a 1-form (connection) as

$$\mathcal{A} = -i\mathcal{A}_\alpha dx^\alpha, \tag{1.2.3}$$

and we express the corresponding (Maxwell) *gauge field* $F_{\mathcal{A}}$ as the 2-form (curvature)

$$F_{\mathcal{A}} = -\frac{i}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta, \tag{1.2.4}$$

where

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{1.2.5}$$

The Higgs field ϕ in (1.2.2) is weakly coupled with the potential field $\mathcal{A}$ via the exterior covariant derivative $D_{\mathcal{A}}$ as follows:

$$D_{\mathcal{A}}\phi = D_\alpha \phi dx^\alpha \text{ where } D_\alpha\phi = \partial_\alpha\phi - i\mathcal{A}_\alpha\phi \text{ and } \alpha = 0, 1, \ldots, d. \tag{1.2.6}$$

We record the Bianchi identity

$$\partial^\gamma F^{\mu\nu} + \partial^\mu F^{\nu\gamma} + \partial^\nu F^{\gamma\mu} = 0, \tag{1.2.7}$$

which is valid for the (curvature) components $F_{\alpha\beta}$ in (1.2.5). For later use, we point out that in the dimension $d = 3$, identity (1.2.7) may be more conveniently expressed in terms of the *dual* gauge field:

$$\tilde{F}_{\mathcal{A}} = -\frac{i}{2}\tilde{F}_{\alpha\beta}\, dx^\alpha \wedge dx^\beta \text{ where } \tilde{F}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$$

as follows:

$$\partial_\beta \tilde{F}^{\alpha\beta} = 0, \qquad \alpha = 0, 1, 2, 3. \tag{1.2.8}$$

Recall that $\varepsilon^{\alpha\beta\gamma\nu}$ denotes the usual Levi-Civita ε-symbol which is totally skew-symmetric with respect to the permutation of indices and is fixed by the condition: $\varepsilon^{0123} = 1$.

In normalized units, the *abelian Maxwell–Higgs* Lagrangean density takes the form

$$\mathcal{L}(\mathcal{A}, \phi) = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} D_\alpha \phi \overline{D^\alpha \phi} - V\left(|\phi|^2\right), \tag{1.2.9}$$

where the *scalar potential* V is taken as the familiar "double-well" potential

$$V(|\phi|^2) = \frac{\lambda}{8}\left(|\phi|^2 - 1\right)^2, \tag{1.2.10}$$

with $\lambda > 0$ a physical parameter. The internal symmetries typical of electromagnetic interactions are expressed by the fact that the fields $\mathcal{A}$ and ϕ are defined only up to the following gauge transformations

$$\phi \to e^{i\omega}\phi, \tag{1.2.11}$$

$$\mathcal{A} \to \mathcal{A} - i d\omega, \tag{1.2.12}$$

for any smooth real function ω over $\mathbb{R}^{1+d}$.

The invariance of $\mathcal{L}$ under the tranformations (1.2.11) and (1.2.12) can be easily verified. In fact, the validity of such invariance property serves as justification for the structure of $\mathcal{L}$ in (1.2.9).

Clearly, the same invariance under (1.2.11) and (1.2.12) is mantained by the corresponding Euler–Lagrange equations

$$D_\mu D^\mu \phi = -2\frac{\partial V}{\partial \overline{\phi}}, \tag{1.2.13}$$

$$\partial_\nu F^{\mu\nu} = J^\mu, \tag{1.2.14}$$

where

$$J^\mu = \frac{i}{2}\left(\overline{\phi} D^\mu \phi - \overline{D^\mu \phi}\phi\right). \tag{1.2.15}$$

Note that J^μ can be considered as the *current* generated by the internal symmetries expressed by (1.2.11) and (1.2.12).

In fact, in view of (1.2.13), J^μ defines a conserved quantity, that is, $\partial_\mu J^\mu = 0$. Furthermore, by identifying

$$\rho = J^0 \text{ and } \mathbf{j} = J^\mu \tag{1.2.16}$$

with the *charge density* and the *current density* respectively, we see that (1.2.8) and (1.2.14) formulate the familiar Maxwell's equations in terms of the *electric field* $\mathbf{E} = (E_j)$ and the *magnetic field* $\mathbf{B} = (B_j)$ specified as follows:

$$E_j = -F^{0j}, \qquad B_j = -\frac{1}{2}\varepsilon^{jkl} F_{kl} \tag{1.2.17}$$

($\varepsilon^{jkl} = \varepsilon^{0jkl}$; $j, k, l = 1, 2, 3$).

In particular, in the absence of the matter field (i.e., $\phi = 0$), $J^\alpha = 0$ and (1.2.14) reduces to Maxwell's equations in a vacuum:

$$\partial_\nu F^{\mu\nu} = 0. \tag{1.2.18}$$

For details see [JT] and [Y1].

The fields $\mathcal{A}$ and ϕ are *not observable* quantities, as they are defined only up to the gauge transformations (1.2.11) and (1.2.12). On the contrary, the electric and magnetic fields (1.2.17) as well as the magnitude $|\phi|$ of the Higgs (matter) field are gauge-independent quantities, and hence observables.

Therefore, from an analytical point of view, we can hope to explicitly solve (1.2.13) and (1.2.14) only in terms of those gauge-invariant quantities.

We shall be interested in obtaining "soliton" configurations, namely, static solutions for (1.2.13) and (1.2.14) carrying *finite* energy. To this end, note that by the Gauss law constraint (i.e., the $\mu = 0$ component of (1.2.14))

$$\partial_j F^{0j} = \frac{i}{2}\left(\overline{\phi} D^0 \phi - \phi \overline{D^0 \phi}\right), \tag{1.2.19}$$

we easily obtain that the *energy density* associated to $\mathcal{L}$

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_\mu\right)} \partial_0 A_\mu + \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)} \partial_0 \phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \overline{\phi}\right)} \partial_0 \overline{\phi} - \mathcal{L} \tag{1.2.20}$$

(in the temporal gauge) takes the form:

$$\mathcal{E} = \frac{1}{2} F_{0j}^2 + \frac{1}{4} F_{jk}^2 + \frac{1}{2}\left|D_0 \phi\right|^2 + \frac{1}{2}\left|D_j \phi\right|^2 + V. \tag{1.2.21}$$

Our interest in *static configurations* is motivated by the fact that in a non-relativistic context, when the dimension is $d = 3$, the Lagrangean density (1.2.9) with the scalar potential V of (1.2.10) has been proposed as a model for superconductivity according to the Ginzburg–Landau theory (cf. [GL]). In this context, ϕ plays the role of the *order parameter*, whose magnitude $|\phi|$ measures the number density of Cooper pairs. Thus, the superconductive state is manifested where $|\phi|$ takes values away from zero. Furthermore, the constant λ involved in the "double-well" scalar potential (1.2.10) defines a relevant physical parameter in this context, as it distinguishes between superconductors of Type I (i.e., $\lambda < 1$) and of Type II (i.e., $\lambda > 1$). By considering a cross section of the superconductive bulk of a material, a "special" situation occurs when we consider bi-dimensional soliton solutions (*vortices*) of (1.2.13) and (1.2.14) at the "critical" coupling value of $\lambda = 1$.

Indeed, from (1.2.21) we see that for $d = 2$ and $\lambda = 1$, the static energy density of the vortex-configurations in the temporal gauge (which allows us to take $A_0 = 0$) coincides with the opposite of the action functional $\mathcal{L}_{\text{static}}$ so that it takes the form,

$$\mathcal{E}_{\text{static}} = -\mathcal{L}_{\text{static}} = \frac{1}{2} F_{12}^2 + \frac{1}{2}|D_1 \phi|^2 + \frac{1}{2}|D_2 \phi|^2 + \frac{1}{8}\left(|\phi|^2 - 1\right)^2. \tag{1.2.22}$$

Thus, if we introduce the operators

$$D_{\pm}\phi = D_1\phi \pm i D_2\phi$$

(to be compared with the $\bar{\partial}$ and ∂ operators of complex functions in a gauge-free setting) and observe that

$$|D_1\phi|^2 + |D_2\phi|^2 = |D_{\pm}\phi|^2 \pm F_{12}|\phi|^2 \mp \varepsilon^{jk}\partial_j J_k, \tag{1.2.23}$$

we arrive at the following useful expression for the *static* energy:

$$\mathcal{E}_{\text{static}} = \frac{1}{2}|D_{\pm}\phi|^2 + \frac{1}{2}\left(F_{12} \pm \frac{1}{2}\left(|\phi|^2 - 1\right)\right)^2 \pm \frac{1}{2}F_{12} \mp \frac{1}{2}\varepsilon^{jk}\partial_j J_k. \tag{1.2.24}$$

Again ε^{jk} is the (bi-dimensional) skew-symmetric ε-symbol ($j, k = 1, 2$), which we can obtain from $\varepsilon^{\alpha\beta\gamma\nu}$ defined above simply by setting $\varepsilon^{jk} = \varepsilon^{0jk3}$.

So, by considering boundary conditions suitable to neglecting the total spatial divergence terms in (1.2.24), we find that *energy minimizer* vortices must satisfy the following *first-order* equations:

$$D_{\pm}\phi = 0, \tag{1.2.25}$$

$$2F_{12} \pm \left(|\phi|^2 - 1\right) = 0, \tag{1.2.26}$$

$$A_0 = 0. \tag{1.2.27}$$

Solutions of (1.2.25)–(1.2.27) are known as the Nielsen–Olesen vortices (cf. [NO]) and are energy minimizers constrained to the class of gauge potential fields with a fixed magnetic flux. We shall see that this corresponds to a "topological" constraint, such to produce "quantization" effects.

Equations (1.2.25)–(1.2.27) were derived by Bogomolnyi in [Bo] as a convenient first-order factorization of the second-order Euler–Lagrange equations (1.2.13) and (1.2.14), where the scalar potential V satisfies (1.2.10) with $\lambda = 1$. Indeed, it is a simple task to check that every solution of (1.2.25)–(1.2.27) also satisfies (1.2.13) and (1.2.14).

Such a reduction property has been observed to occur in quite a variety of models in gauge field theory, when the physical parameters are specified according to an appropriate "critical" coupling. The first instance of such an occurance has been observed in the non-abelian context for the pure Yang–Mills model (cf. [JT]). In this case, energy minimizers (within a topological class) give rise to *instantons* and correspond to selfdual (or antiselfdual) connections, as discussed in Section 1.3.5.

By analogy, it has become a custom to refer to the reduced first-order equations as the *selfdual (antiselfdual) equations*. We can reveal the selfdual/antiselfdual character of (1.2.25) if we express it in the form:

$$D_j\phi = \mp i\varepsilon_{jk}D_k\phi. \tag{1.2.28}$$

However, before engaging with non-abelian (selfdual) gauge field theories, let us see how a similar reduction procedure can also be attained when we enrich the electrodynamical properties of the theory by including the Chern–Simons term.

The Chern–Simons theory is a planar theory (i.e., $d = 2$, or more general d is even) that enjoys several favorable physical properties not attainable through the "conventional" Maxwell electrodynamics. For instance, we shall see that Maxwell–Chern–Simons vortex-configurations carry both electrical and magnetic charge, in contrast with the conventional Maxwell electrodynamics, that only yields to electrically neutral Ginzburg–Landau vortices. This and other important aspects of (relativistic and non-relativistic) Chern–Simons theories are widely discussed by Dunne in [D1], in relations to their relevance in high critical temperature superconductivity, the quantum Hall effect, conformal field theory and planar condensed matter physics.

In the abelian context, the Chern–Simons Lagrangean density $\mathcal{L}_{cs}$ is assigned in $\mathbb{R}^{1+2}$ in terms of the potential field $\mathcal{A} = -i\mathcal{A}_\alpha dx^\alpha,\ \alpha = 0, 1, 2$ as

$$\mathcal{L}_{cs}(\mathcal{A}) = \frac{1}{4}\varepsilon^{\alpha\beta\gamma}\mathcal{A}_\alpha F_{\beta\gamma}, \tag{1.2.29}$$

where again $\varepsilon^{\alpha\beta\gamma}$ denotes the totally skew-symmetric pseudotensor fixed by setting $\varepsilon^{012} = 1$.

The structure of the Chern–Simons Lagrangean $\mathcal{L}_{cs}$ may be justified on the basis of a reduction argument from the 4-dimensional Yang–Mills equations to dimension $d = 2$ (see [D1] for details). In this respect, observe that $\mathcal{L}_{cs}$ corresponds to the action functional for the (trivial) equation

$$F_{\alpha\beta} = 0 \tag{1.2.30}$$

since we have

$$\frac{\partial \mathcal{L}_{cs}}{\partial A_\mu} = \frac{1}{4}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta}. \tag{1.2.31}$$

We remark the interesting fact that although $\mathcal{L}_{cs}$ is not gauge-invariant,

$$\mathcal{L}_{cs}(\mathcal{A} - id\omega) = \mathcal{L}_{cs}(\mathcal{A}) + \frac{1}{2}\partial_\mu\left(\omega\varepsilon^{\mu\alpha\beta}\partial_\alpha\mathcal{A}_\beta\right);$$

the corresponding Euler–Lagrange equation (1.2.30) is gauge-invariant, and for this reason, $\mathcal{L}_{cs}$ is an admissible Lagrangean in the context of gauge field theory.

As a first example, we describe a theory proposed by Jackiw–Weinberg [JW] and Hong–Kim–Pac [HKP] in which the electrodynamics of the system is governed solely by the Chern–Simons Lagrangean. The corresponding Chern–Simons–Higgs Lagrangean density takes the form:

$$\mathcal{L}(\mathcal{A}, \phi) = -\frac{k}{4}\varepsilon^{\alpha\beta\mu}\mathcal{A}_\alpha F_{\beta\mu} + D_\alpha\phi\overline{D^\alpha\phi} - V\left(|\phi|^2\right), \tag{1.2.32}$$

where the (dimensionless) coupling constant $k > 0$ measures the strength of the Chern–Simons term, which we shall refer to as the Chern–Simons parameter.

The Euler–Lagrange equations relative to (1.2.32) are expressed as

$$D_\alpha D^\alpha \phi = -\tfrac{\partial V}{\partial \overline{\phi}}, \tag{1.2.33}$$

$$\tfrac{k}{2}\varepsilon^{\mu\alpha\beta} F_{\alpha\beta} = J^\mu, \tag{1.2.34}$$

where

$$J^\mu = i\left(\overline{\phi} D^\mu \phi - \phi \overline{D^\mu \phi}\right). \tag{1.2.35}$$

Again, J^μ can be considered as the conserved current for the system, with $\rho = J^0$ the charge density and $\mathbf{j} = J^k$ the current density. As before, using the Gauss law constraint obtained from the $\mu = 0$ component of (1.2.34),

$$kF_{12} = i\left(\overline{\phi} D^0 \phi - \phi \overline{D^0 \phi}\right), \tag{1.2.36}$$

we easily deduce that for the Lagrangean density (1.2.32), the associate *energy density* takes the expression

$$\mathcal{E} = |D_0\phi|^2 + |D_j\phi|^2 + V\left(|\phi|^2\right)$$

(provided we neglect the total divergence term). Since $\mathcal{L}_{cs}$ is independent of the Minkowski metric, we check that indeed, it does not contribute to the energy momentum tensor.

Thus (pure) Chern–Simons vortices will correspond to solutions for (1.2.33) and (1.2.34) independent of the x^0-variable and with finite (static) energy. Note that, the Gauss law constraint (1.2.36) for the time-independent case reduces to

$$kF_{12} = 2A_0|\phi|^2 = J_0. \tag{1.2.37}$$

Identity (1.2.37) can be used together with the identity

$$|D_1\phi|^2 + |D_2\phi|^2 = |D_\pm\phi|^2 \pm F_{12}|\phi|^2 \mp \frac{\varepsilon^{jk}}{2}\partial_j J_k \tag{1.2.38}$$

(the equivalent of (1.2.23)) to obtain the gauge-invariant part of the energy density relative to vortex configurations as

$$\begin{aligned}\mathcal{E}_{\text{static}} &= |A_0\phi|^2 + |D_\pm\phi|^2 \pm F_{12}|\phi|^2 + V\left(|\phi|^2\right) \mp \frac{\varepsilon^{jk}}{2}\partial_j J_k \\ &= |D_\pm\phi|^2 \pm \frac{2}{k}A_0|\phi|^4 + A_0^2|\phi|^2 + V\left(|\phi|^2\right) \mp \frac{\varepsilon^{jk}}{2}\partial_j J_k.\end{aligned} \tag{1.2.39}$$

As for the abelian Maxwell–Higgs model, we can operate on a suitable choice of the scalar potential V in order to complete the square in (1.2.39) and identify a set of first-order (selfdual) equations that "factorizes" (1.2.33) and (1.2.34). To accomplish this goal and account for some physical consistency, Jackiw–Weinberg in [JW] and Hong–Kim–Pac in [HKP] proposed the *triple-well* potential

$$V\left(|\phi|^2\right) = \frac{1}{k^2}|\phi|^2\left(|\phi|^2 - \nu^2\right)^2, \tag{1.2.40}$$

with the mass-scale symmetry-breaking parameter ν^2. For the abelian Maxwell–Higgs model (1.2.9) we have adopted the normalization $\nu^2 = 1$. Also notice the explicit dependence of the (selfdual) scalar potential (1.2.40) on the Chern–Simons parameter k. Therefore as for the abelian Maxwell–Higgs model, we see that a selfdual regime is reached when there is a balance between the strength of the electromagnetic term (Maxwell or Chern–Simons) and the potential term.

For the Chern–Simons 6th-order model proposed in [JW] and [HKP],

$$\mathcal{L}(\mathcal{A}, \phi) = -\frac{k}{4}\varepsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} + D_\alpha\phi\overline{D^\alpha\phi} - \frac{1}{k^2}|\phi|^2\left(|\phi|^2 - \nu^2\right)^2, \quad (1.2.41)$$

the (static) energy density of vortex-configurations takes the form

$$\mathcal{E}_{\text{static}} = |D_\pm\phi|^2 + |\phi|^2\left(A_0 \pm \frac{1}{k}\left(|\phi|^2 - \nu^2\right)\right)^2 \pm \nu^2 F_{12} \mp \frac{\varepsilon^{jk}}{2}\partial_j J_k. \quad (1.2.42)$$

Consequently, by considering suitable boundary conditions that allow one to neglect the total spatial divergence term in (1.2.42), we arrive at the following first-order equations to be satisfied by energy minimizers (at fixed flux) Chern–Simons vortex solutions, together with the Gauss-law constraint (1.2.37)

$$D_\pm\phi = 0, \quad (1.2.43)$$

$$|\phi|(A_0 \pm \frac{1}{k}\left(|\phi|^2 - \nu^2\right) = 0. \quad (1.2.44)$$

We can arrange (1.2.37), (1.2.43), and (1.2.44) more conveniently in the following equivalent set of *selfdual equations*:

$$\begin{cases} D_\pm\phi = 0, \\ F_{12} = \pm\frac{2}{k^2}|\phi|^2\left(\nu^2 - |\phi|^2\right), \\ 2A_0|\phi|^2 = kF_{12}. \end{cases} \quad (1.2.45)$$

Again (1.2.45) represents a "factorization" of the second-order Euler–Lagrange equations (1.2.33) and (1.2.34). Indeed, one can easily check that a solution of (1.2.45) also verifies (1.2.33) and (1.2.34) with V specified by (1.2.40).

In particular observe that away from the zeros of ϕ, the A_0-component of the potential field is determined simply by the identity:

$$A_0 = \pm\frac{1}{k}\left(\nu^2 - |\phi|^2\right) \quad (1.2.46)$$

(see (1.2.44)). Thus, the selfdual vortex solution is completely identified by the components (A_1, A_2, ϕ), satisfying the first two equations in (1.2.45).

It is interesting to note that the Chern–Simons energy density takes a similar structure (*i.e., sum of the quadratic terms plus the spatial divergence terms*) for the time-dependent solutions of (1.2.33) and (1.2.34), provided we specify the scalar potential V as in (1.2.40).

In fact, the (non-static) Gauss law constraint (1.2.36) may be used together with (1.2.38) to deduce the following expression for the energy density:

$$\mathcal{E}_{cs} = \left| D^0\phi \mp \frac{i}{k}\phi \left(|\phi|^2 - \nu^2 \right) \right|^2 + |D_\pm \phi|^2 \pm \frac{\nu^2}{k} J^0 \mp \frac{\varepsilon^{jk}}{2} \partial_j J_k. \quad (1.2.47)$$

As noted above, for fixed magnetic flux (see (1.2.37)), energy minimizers may be identified as solutions of the first-order equations

$$D_\pm \phi = 0, \quad (1.2.48)$$

$$D^0\phi = \pm \frac{i}{k}\phi \left(|\phi|^2 - \nu^2 \right), \quad (1.2.49)$$

to be satisfied in addition to the Gauss law constraint (1.2.36). Equivalently, by inserting equation (1.2.49) into (1.2.36) for the time-dependent case, we obtain the following Chern–Simons selfdual equations:

$$\begin{cases} D_\pm \phi = 0, \\ F_{12} = \pm \frac{2}{k^2} |\phi|^2 \left(\nu^2 - |\phi|^2 \right), \\ D^0\phi = \pm \frac{i}{k}\phi \left(|\phi|^2 - \nu^2 \right). \end{cases} \quad (1.2.50)$$

A soliton-like solution of (1.2.50) may be constructed out of a solution $(A_0, A_1, A_2, \phi)_{\text{static}}$, of the static selfdual Chern–Simons equations (1.2.45), simply by letting

$$\begin{aligned} &\phi = e^{\pm \frac{i\omega^2 x_0}{k}} \phi_{\text{static}}, \\ &A_j = (A_j)_{\text{static}}, \qquad A_0 = \pm \frac{1}{k} \left(\nu^2 + \omega^2 - |\phi_{\text{static}}|^2 \right), \end{aligned} \quad (1.2.51)$$

for any constant $\omega \in \mathbb{R}$.

In view of (1.2.46) the soliton-like solution above reduces to the selfdual vortex $(A_0, A_1, A_2, \phi)_{\text{static}}$, when $\omega = 0$.

Next we see how an analogous selfdual reduction property remains valid by considering a full Maxwell–Chern–Simons–Higgs theory (MCSH-theory), that also includes a neutral scalar field.

We shall discuss a model proposed by Lee–Lee–Min [LLM] with the purpose of unifying the abelian Maxwell–Higgs and Chern–Simons-Higgs theories considered above.

To this purpose, it is convenient to introduce the explicit dependence of the theory in terms of the electric charge q (previously normalized to 1) so that the (dimensionless) Chern–Simons parameter is expressed as

$$k = \frac{\sigma}{q^2}, \quad (1.2.52)$$

with σ having the dimension of mass.

The MCSH-theory proposed in [LLM] is formulated by means of the Lagrangean density

$$\begin{aligned}\mathcal{L}(\mathcal{A},\phi,N) = &-\frac{1}{4q^2}F_{\alpha\beta}F^{\alpha\beta} - \frac{\sigma}{4q^2}\varepsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + D_\alpha\phi\overline{D^\alpha\phi}\\ &+\frac{1}{2q^2}\partial_\alpha N\partial^\alpha N - V\left(|\phi|^2, N\right),\end{aligned} \tag{1.2.53}$$

where N is a *neutral scalar* field, and the scalar potential V takes the form

$$V\left(|\phi|^2, N\right) = |\phi|^2\left(N - \frac{q^2\nu^2}{\sigma}\right)^2 + \frac{q^2}{2}\left(|\phi|^2 - \frac{\sigma}{q^2}N\right)^2, \tag{1.2.54}$$

with the mass scale parameter ν^2 to be considered as a symmetry-breaking parameter.

Note that formally, we may recover the abelian Maxwell–Higgs Lagrangean density (1.2.9) out of (1.2.53) by letting $\sigma \to 0$ while keeping q fixed. In this way the Chern–Simons term drops out while the neutral scalar field N needs to be fixed according to the relation

$$\frac{\sigma N}{q^2} = \nu^2.$$

Then (1.2.53) reduces to

$$\mathcal{L}^{AH}(\mathcal{A},\phi) = -\frac{1}{4q^2}F_{\alpha\beta}F^{\alpha\beta} + D_\alpha\phi\overline{D^\alpha\phi} - \frac{q^2}{2}\left(|\phi|^2 - \nu^2\right)^2, \tag{1.2.55}$$

which gives exactly the selfdual abelian Maxwell–Higgs Lagrangean density with all relevant physical parameters.

On the other hand, if we let $\sigma \to +\infty$, $q \to +\infty$ while keeping fixed the Chern–Simons constant given by the ratio (1.2.52), we again formally see that both the Maxwell term and the kinetic term relative to the neutral field drop out, while N must be fixed according to the relation $\frac{\sigma}{q^2}N = |\phi|^2$. Thus at the limit, $\mathcal{L}$ in (1.2.53) takes exactly the form of the Chern–Simons Lagrangean (1.2.32).

It is in this sense that we say that the MCSH-Lagrangean in (1.2.53) formulates a *unified* theory between the abelian Maxwell–Higgs and Chern–Simons models.

The "formal" limits taken above can be shown to hold rigorously along vortex configurations (see [RT1]).

Let us now describe the selfdual structure of the MCSH-Lagrangean represented by (1.2.53) and (1.2.54). First of all, notice that the corresponding Euler–Lagrange equations complete those in (1.2.34) and (1.2.33) as we have:

$$\begin{cases}\frac{1}{q^2}\partial_\alpha\partial^\alpha N = -\frac{\partial V}{\partial N}\left(|\phi|^2, N\right),\\ D_\alpha D^\alpha\phi = -\frac{\partial V}{\partial\overline{\phi}}\left(|\phi|^2, N\right),\\ \frac{1}{q^2}\partial_\nu F^{\mu\nu} + \frac{k}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta} = J^\mu,\end{cases} \tag{1.2.56}$$

with k in (1.2.52) and J^μ the *current* defined in (1.2.35).

As above, the $\mu = 0$ component of the last equation in (1.2.56) expresses the Gauss law constraint for the given system, which is given as follows:

$$\frac{1}{q^2}\partial_j F^{0j} + kF_{12} = J^0 = i\left(\overline{\phi}D^0\phi - \phi\overline{D^0\phi}\right). \tag{1.2.57}$$

As before, we shall take advantage of (1.2.57) and (1.2.38) in order to express the following MCSH-energy density:

$$\begin{aligned}\mathcal{E} = {} & \frac{1}{2q^2}|F_{0j}|^2 + \frac{1}{2q^2}|F_{12}|^2 + |D_0\phi|^2 + |D_1\phi|^2 + |D_2\phi|^2 + \frac{1}{2q^2}|\partial_0 N|^2 \\ & + \frac{1}{2q^2}|\nabla N|^2 + |\phi|^2\left(N - \frac{q^2}{\sigma}\nu^2\right)^2 + \frac{q^2}{2}\left(|\phi|^2 - \frac{\sigma}{q^2}N\right)^2.\end{aligned} \tag{1.2.58}$$

(neglecting a total spatial divergence term) in the more convenient form:

$$\begin{aligned}\mathcal{E} = {} & \frac{1}{2q^2}\left|F^{0j} \pm \partial_j N\right|^2 + \frac{1}{2q^2}|\partial_0 N|^2 + |D_\pm\phi|^2 \\ & + \frac{1}{2q^2}\left(F_{12} \pm q^2\left(|\phi|^2 - \frac{\sigma}{q^2}N\right)\right)^2 + \left|D_0\phi \mp i\phi\left(N - \frac{\nu^2}{k}\right)\right|^2 \\ & \pm \frac{\nu^2}{k}J^0 \mp \partial_j\left(\frac{\varepsilon^{jk}}{2}J_k + \frac{1}{q^2}NF_{0j}\right).\end{aligned} \tag{1.2.59}$$

Therefore, with the help of suitable boundary conditions that allow one to neglect the last total spatial divergence term in (1.2.59), the minimal MCSH-energy for fixed flux, is saturated by the solutions of the following *selfdual equations*:

$$\begin{cases} D_\pm\phi = 0, \\ F^{0j} \pm \partial_j N = 0, \\ F_{12} \pm q^2\left(|\phi|^2 - kN\right) = 0, \\ D_0\phi \mp i\phi\left(N - \frac{\nu^2}{k}\right) = 0, \\ \partial_0 N = 0, \end{cases} \tag{1.2.60}$$

with k given in (1.2.52).

Once more, equations (1.2.60) supplemented with the Gauss law constraint (1.2.57) identify a first-order factorization of the second-order Euler–Lagrange equations (1.2.56).

In particular, selfdual MCSH-vortices will be obtained by considering solutions of (1.2.60) independent of the x^0-variable. In this situation, the corresponding equations simplify considerably, since the last equation is automatically satisfied. We can satisfy the second and fourth equation in (1.2.60) simply by setting:

$$A_0 = \pm\left(\frac{\nu^2}{k} - N\right). \tag{1.2.61}$$

Furthermore, in the x^0-independent case the Gauss law constraint (1.2.57) takes the form

$$\frac{1}{q^2}\Delta A_0 + kF_{12} = 2A_0|\phi|^2. \tag{1.2.62}$$

Therefore, we can combine (1.2.61), (1.2.62), and the third equation in (1.2.60) to see that a selfdual MCSH-vortex is completely determined in terms of the components (A_1, A_2, ϕ, N) satisfying:

$$\begin{cases} D_\pm \phi = 0, \\ F_{12} = \pm q^2 \left(kN - |\phi|^2\right), \\ -\frac{1}{q^2}\Delta N = 2|\phi|^2 \left(\frac{\nu^2}{k} - N\right) + kq^2 \left(|\phi|^2 - kN\right). \end{cases} \tag{1.2.63}$$

Exactly as before, we see that from a solution $(A_1, A_2, \phi, N)_{\text{static}}$ of (1.2.63), we may obtain a soliton-like solution of (1.2.60) by letting:

$$\phi = e^{\pm \frac{i\omega^2 x_0}{k}} \phi_{\text{static}}, \qquad A_j = (A_j)_{\text{static}}, \; j = 1, 2,$$
$$N = N_{\text{static}}, \qquad \text{and} \qquad A_0 = \pm\left(\frac{1}{k}\left(\nu^2 + \omega^2\right) - N_{\text{static}}\right),$$

for every $\omega \in \mathbb{R}$.

Analogous selfdual reduction procedures are known to hold also for non-relativistic versions of the Maxwell and Chern–Simons theories described above, where roughly speaking, the covariant derivative $D_0\phi$ of the Higgs field only enters linearly into the Lagrangean density. We refer to [D1] for a detailed discussion in this direction.

Already from those first examples, we can remark on some interesting features common to all selfdual equations discussed so far. Firstly, all of them include the self-dual/antiselfdual equation

$$D_\pm \phi = 0, \tag{1.2.64}$$

which will play a crucial role in the analysis that follows. For the moment, let us mention that (1.2.64) may be viewed as a gauge-invariant version of the Cauchy–Riemann equation, and in fact it implies an holomorphic-type behavior for the Higgs field ϕ (respectively $\overline{\phi}$) up to gauge tranformations (cf. [JT]). Note also that except for (1.2.64), the remaining (static) selfdual equations involve only gauge-invariant quantities (i.e., F_{12}, $|\phi|^2$, and when present, the neutral scalar field N). Therefore, one may hope to find an appropriate gauge transformation according to which the full set of selfdual equations take the most convenient expression from the analytical point of view.

This goal was attained first by Taubes (see [Ta1], [Ta2], and [JT]) who successfully handled the selfdual abelian Maxwell–Higgs model. It is one of our purposes to show that, in fact, Taubes' approach works equally well for the Chern–Simons models discussed above and, more generally, for the non-abelian theories of next section.

But before treating non-abelian gauge field theories, we wish to mention that while Taubes' approach has furnished a complete characterization of Ginzburg–Landau vortices in the selfdual regime, in recent years much progress has also been made away from the selfdual regime (i.e., $\lambda \neq 1$ in (1.2.10)). In fact, a much better understanding of Ginzburg–Landau vortex configurations now exists in dimension $d = 2$ and $d = 3$, also in relation to their dynamical properties. In this respect, see for example: [AM], [AB], [ABG], [JMS], [BeR], [JiR], [ABP], [BPT], [BBH], [BBO], [BOS], [BR], [CHO], [CRS], [DGP], [E], [J1], [J2], [JS1], [JS2], [Lin1], [Lin2], [Lin3], [LR1], [LR2], [LR3], [MSZ], [PiR], [Riv], [RuS], [Sa], [SS1], [SS2], [SS3], [SS4], [SS5], [SS6], [SS7], [Se1], [Se2], [Se3], and [Spi]. Such a remarkable understanding of the Ginzburg–Landau model has also prompted to undertake a similar approach to the 6th-order Chern–Simons model away from the selfdual regime; contributions in this direction can be found in [HaK], [KS1], and [KS2].

1.3 Non-abelian gauge field theories

1.3.1 Preliminaries

In this section we discuss examples of gauge field theories describing physical interactions other than the electromagnetic ones treated in the previous section.

Such theories are formulated within the mathematical framework of (non-abelian) gauge theory and are specified according to (a representation of) an assigned gauge group G. The gauge group G is given by a real Lie group, usually compact and connected, which acts over a finite-dimensional (real or complex) linear space $\mathbb{L}$. The corresponding representation,

$$\rho : G \to \text{Aut}\,(\mathbb{L}), \tag{1.3.1}$$

will be used to describe the internal (local) symmetries relative to the theory. For this reason, we shall refer to $\mathbb{L}$ as the *internal symmetry* space.

By the usual operation of "derivation," we can exchange information between the group G and its (real) Lie algebra $\mathcal{G}$. In this way, ρ in (1.3.1) induces a representation of $\mathcal{G}$ on $\mathbb{L}$ which we shall denote in the same way. We refer to [Ca], [Fe], and [Hu] for the details.

In most cases we can use (1.3.1) to identify G with an embedded subgroup of the Lie group of (square) real invertible matrices:

$$GL(n, \mathbb{R}) = \{n \times n \text{ real matrix } A : \ \det A \neq 0\}\,. \tag{1.3.2}$$

So, for most purposes, it is convenient to think of G as a matrix group.

In this respect, note that G could be represented also by complex square matrices, namely, elements of the group:

$$GL(n, \mathbb{C}) = \{A : \ n \times n \text{ invertible complex matrix}\}\,; \tag{1.3.3}$$

but in this case it is understood that G (being a real group) is considered a subgroup of $GL(2n, \mathbb{R})$.

1.3.2 The adjoint representation and some examples

In this section, we describe the *adjoint representation* for a Lie group G. This will help establish ideas and will give us a chance to review a relevant group representation for the physical applications. More precisely, the adjoint representation concerns the situation where $\mathbb{L} = \mathcal{G}$; namely, the linear (symmetry) space coincides with the Lie algebra of $\mathcal{G}$, and

$$Ad : G \longrightarrow \text{Aut}\,\mathcal{G} \tag{1.3.4}$$

is defined as follows.

Let $e \in G$ be the *unit* element of G, *i.e.*, $ge = g = eg\ \forall g \in G$. And for $g \in G$, define the inner automorphism on G as

$$\begin{aligned} I_g : G &\longrightarrow G \\ a &\longrightarrow gag^{-1}. \end{aligned}$$

Then

$$Ad(g) = d(I_g)|_e \in \text{Aut}\,\mathcal{G}, \tag{1.3.5}$$

where d denotes the usual differentiation of smooth functions on manifolds. The induced adjoint representation over the Lie algebra $\mathcal{G}$,

$$ad : \mathcal{G} \longrightarrow \text{End}\,\mathcal{G} \quad \text{with } ad = d(Ad)|_e, \tag{1.3.6}$$

can be shown to act simply by the Lie bracket [,] on $\mathcal{G}$, namely,

$$ad(A) = [A, \cdot] \tag{1.3.7}$$

(see [Fe]).

The (real) adjoint representation introduced above can be extended in a natural way over the complexification of the (real) Lie algebra $\mathcal{G}$ as

$$\mathcal{G}_{\mathbb{C}} = \mathcal{G} \otimes_{\mathbb{R}} \mathbb{C}, \tag{1.3.8}$$

which defines a *complex* Lie algebra equipped with the induced Lie bracket. $\mathcal{G}_{\mathbb{C}}$ preserves the structural properties of G and we may consider the extended maps

$$Ad : G \longrightarrow \text{Aut}\,(\mathcal{G}_{\mathbb{C}}) \tag{1.3.9}$$

and

$$\begin{aligned} ad : \mathcal{G}_{\mathbb{C}} &\longrightarrow \text{End}\,(\mathcal{G}_{\mathbb{C}}) \\ A &\longrightarrow [A, \cdot], \end{aligned} \tag{1.3.10}$$

where for $g \in G$, the restrictions $Ad(g)|_{\mathcal{G}}$ and $ad|_{\mathcal{G}}$ recover, respectively, the adjoint representation (1.3.5) and (1.3.7).

We shall consider both real and complex adjoint representations. In literature, the complex adjoint representation is also referred to as the conjugate representation of G.

In the context of the adjoint representation, it is interesting to consider the *Killing form*

$$k : \mathcal{G}_{\mathbb{C}} \times \mathcal{G}_{\mathbb{C}} \longrightarrow \mathbb{C},$$

given by the symmetric bilinear form defined as

$$k(A, B) = \text{tr}(ad(A)ad(B)), \tag{1.3.11}$$

where $A, B \in \mathcal{G}_{\mathbb{C}}$ and tr denote the trace of elements in End$(\mathcal{G}_{\mathbb{C}})$. The Killing form is invariant under the adjoint transformation of (1.3.9) (and hence (1.3.5)), in the sense that, for $A, B \in \mathcal{G}_{\mathbb{C}}$:

$$k(Ad(g)(A), Ad(g)(B)) = k(A, B),\ \forall g \in G; \tag{1.3.12}$$

and $k|_{\mathcal{G}\times\mathcal{G}} \in \mathbb{R}$. For later use, we record the identity

$$k\left([A, B], C\right) = k\left(A, [B, C]\right), \tag{1.3.13}$$

which follows from elementary properties of Lie brackets and traces. Note that the Killing form vanishes identically when G is *abelian*, i.e., every element of $\mathcal{G}$ (and thus of $\mathcal{G}_{\mathbb{C}}$) commute: $[A, B] = 0,\ \forall A, B \in \mathcal{G}_{\mathbb{C}}$. On the other hand, it is also of interest to analyze the situation where k is *non-degenerate*, in which case the relative group is said to be a *semisimple group*.

It is possible to show that (see [Fe]):

Theorem 1.3.1 *If G is connected and semisimple, then G is compact if and only if $k|_{\mathcal{G}\times\mathcal{G}}$ is negative definite.*

In the context of Theorem 1.3.1, the Killing form is used to provide $\mathcal{G}$ with a Riemannian structure. In the sequel, we shall explore its role in the Cartan–Weyl decomposition of $\mathcal{G}_{\mathbb{C}}$.

Next, let us see how the adjoint representation operates over a matrix group. For this purpose, recall that the Lie algebra associated to $GL(n, \mathbb{R})$ is given by

$$gl(n, \mathbb{R}) = \{n \times n \text{ real matrices}\}, \tag{1.3.14}$$

whose complexification

$$(gl(n, \mathbb{R}))_{\mathbb{C}} = gl(n, \mathbb{C}) = \{n \times n \text{ complex matrices}\} \tag{1.3.15}$$

just corresponds to the Lie algebra of the *complex* Lie group $GL(n, \mathbb{C})$. If G is a (embedded) subgroup of $GL(n, \mathbb{R})$, then it is not difficult to see that: if $g \in G \subset GL(n, \mathbb{R})$ and $A \in \mathcal{G} \subset gl(n, \mathbb{R})$ (or $A \in \mathcal{G}_{\mathbb{C}} \subset gl(n, \mathbb{C})$), then

$$Ad(g)A = gAg^{-1} \in \mathcal{G} \text{ (or } \mathcal{G}_{\mathbb{C}}), \tag{1.3.16}$$

where we use the usual matrix multiplication.

In fact, through an abuse of notation, the expression (1.3.16) can be adopted to denote, in general, the (real or complex) adjoint representation for any group.

Next we recall some examples of matrix groups that will enter as gauge groups in the gauge field theories discussed below.

Special Orthogonal Group:

$$SO(n) = \left\{A \in GL(n, \mathbb{R}) : \; A\,{}^tA = Id, \; \det A = 1\right\} \tag{1.3.17}$$

where ${}^tA$ is the transpose of A, and its associated Lie algebra is

$$so(n) = \left\{A \in gl(n, \mathbb{R}) : \; A = -\,{}^tA\right\}. \tag{1.3.18}$$

Special Unitary Group:
The complex counterpart of $SO(n)$ is given by,

$$SU(n) = \left\{A \in GL(n, \mathbb{C}) : \; AA^{\dagger} = Id, \; \det A = 1\right\} \tag{1.3.19}$$

where $A^{\dagger} = \overline{{}^tA}$ is the Hermitian conjugate of A, and its associated Lie algebra is

$$su(n) = \left\{A \in gl(n, \mathbb{C}) : \; A = -A^{\dagger}, \; \mathrm{tr} A = 0\right\}. \tag{1.3.20}$$

In particular note that

$$\dim_{\mathbb{R}} su(n) = n^2 - 1, \tag{1.3.21}$$

and the *complexification* of $su(n)$ is given by

$$(su(n))_{\mathbb{C}} = sl(n, \mathbb{C}) = \{A \in gl(n, \mathbb{C}) : \; \mathrm{tr}\, A = 0\}. \tag{1.3.22}$$

We recall that $SU(n)$ defines a *compact, connected* subgroup of the unitary group

$$U(n) = \left\{A \in GL(n, \mathbb{C}) : \; AA^{\dagger} = Id\right\} \tag{1.3.23}$$

and also defines a *semisimple* group for $n \geq 2$ (cf. [Ca], [Fe]). While for $n = 1$, $U(1) = \{z \in \mathbb{C} : \; |z| = 1\}$ defines an *abelian* compact group which we can identify with $SO(2)$.

In fact, every element of $SO(2)$ can be described by a pair of complex numbers $\{z, \omega\}$ such that $\omega = iz$ and $|z| = 1$. We thus see that the projection map from $\mathbb{C}^2$ into $\mathbb{C}$ defines a Lie group homeomorphism between $SO(2)$ and $U(1)$.

Note that $U(1)$ acts on $\mathbb{C}$ by multiplication by a unitary complex number.

In concluding this section we return to the role of differentiation as a means to transfer information from G over (its infinitesimal expression) $\mathcal{G}$, as we have seen already from (1.3.5) to (1.3.6). As is well-known, this procedure may be reversed by means of the exponential map. For a matrix group $G \subset GL(n, \mathbb{R})$ (or $G \subset GL(n, \mathbb{C})$) with Lie algebra $\mathcal{G} \subset gl(n, \mathbb{R})$ (or $\mathcal{G} \subset gl(2n, \mathbb{R})$), this is simply described by the *exponential of a matrix*:

$$A \in \mathcal{G} \longrightarrow e^A = \sum_{k=1}^{+\infty} \frac{A^k}{k!} \in G. \tag{1.3.24}$$

1.3.3 Gauge field theories

We return to the description of a gauge theory over a (real) Lie group G, acting over the symmetry space $\mathbb{L}$ with the corresponding representation ρ in (1.3.1).

From now on we shall assume G to be compact.

In this situation, it is always possible to define an inner product over $\mathbb{L}$ and $\mathcal{G}$ invariant under the transformation of $\rho(g)$ and $Ad(g)$, respectively, $\forall g \in G$ (cf. [Ca]). We denote it by $(\cdot, \cdot)$ regardless of whether it pertains to $\mathbb{L}$ or $\mathcal{G}$, so that no confusion should arise. And we let $|\cdot|$ denote the associated norm.

A gauge field theory with gauge group G is formulated in terms of the following dynamical variables:

the gauge potential $\mathcal{A}$: a connection over the principal bundle P with structure group G,

the Higgs (matter) field ϕ: a smooth section of the associated bundle E with fiber $\mathbb{L}$;

(cf. [JT], [Tra], and [Jo]).

For the purpose of these notes, it suffices to limit our attention to gauge theories where both the principle and the associated bundle are trivial; although it should be mentioned that non-trivial bundles do occur in physical literature.

More precisely we take,

$$P = \mathbb{R}^{1+d} \times G, \qquad E = \mathbb{R}^{1+d} \times \mathbb{L},$$

with $(\mathbb{R}^{1+d}, g)$ the Minkowski space and $g = \text{diag}\,(1, -1, \dots, -1)$. Hence, the potential field $\mathcal{A}$ is the globally defined $\mathcal{G}$-valued 1-form

$$\mathcal{A} = A_\alpha dx^\alpha, \qquad A_\alpha = A_\alpha(x) \in \mathcal{G} \quad \alpha = 0, 1, \dots, d, \tag{1.3.25}$$

and the Higgs field ϕ is the smooth $\mathbb{L}$-valued function

$$\phi : \mathbb{R}^{1+d} \longrightarrow \mathbb{L}. \tag{1.3.26}$$

In turn, if we let $D_{\mathcal{A}}$ denote the (exterior) covariant derivative associated to the connection $\mathcal{A}$ acting on $\mathcal{G}$-valued forms, then we obtain the *gauge field* $F_{\mathcal{A}}$ as the curvature 2-form corresponding to $\mathcal{A}$:

$$F_{\mathcal{A}} = D_{\mathcal{A}}\mathcal{A} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \frac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta, \tag{1.3.27}$$

with

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + \left[A_\alpha, A_\beta\right], \tag{1.3.28}$$

$\alpha, \beta = 0, 1, \dots, d$.

By means of the (induced) representation of ρ on $\mathcal{G}$, we can also consider the covariant derivative $D_{\mathcal{A}}$ acting over $\mathbb{L}$-valued forms as follows:

$$D_{\mathcal{A}}\omega = d\omega + \rho(\mathcal{A}) \wedge \omega.$$

In this way, we can (weakly) couple the Higgs field ϕ to the potential $\mathcal{A}$ as follows:

$$D_{\mathcal{A}}\phi = D_\alpha\phi dx^\alpha \text{ with } D_\alpha\phi = \partial_\alpha\phi + \rho(A_\alpha)\phi, \tag{1.3.29}$$

$\alpha = 0, 1, \ldots, d$.

Observe that when we represent G according to the (real or complex) adjoint representation, the components of the covariant derivative of ϕ reduce to the expression

$$D_\alpha\phi = \partial_\alpha\phi + [A_\alpha, \phi], \tag{1.3.30}$$

$\alpha = 0, 1, \ldots, d$.

A (relativistic) gauge field theory (in normalized units) is formulated by means of a Lagrangean density of the form

$$\mathcal{L}(\mathcal{A}, \phi) = -\frac{1}{4}\left(F_{\alpha\beta}, F^{\alpha\beta}\right) + \frac{1}{2}\left(D_\alpha\phi, D^\alpha\phi\right) - V, \tag{1.3.31}$$

with scalar potential V typically assigned with dependence on $|\phi|^2 = (\phi, \phi)$.

The internal symmetries of the theory are now expressed by the invariance of $\mathcal{L}$ under the gauge transformations

$$A \longrightarrow A_g = Ad(g)\mathcal{A} + gdg^{-1}, \tag{1.3.32}$$

$$\phi \longrightarrow \phi_g = \rho(g)\phi, \tag{1.3.33}$$

for any given smooth *gauge map*

$$g : \mathbb{R}^{1+d} \longrightarrow G. \tag{1.3.34}$$

Indeed, it is not difficult to verify the following (covariant) transformation rules:

$$F_{A_g} = Ad(g)F_{\mathcal{A}}, \qquad D_{A_g}\phi_g = \rho(g)D_{\mathcal{A}}\phi. \tag{1.3.35}$$

Consequently, the invariance of the inner product (relative to $\mathbb{L}$ or $\mathcal{G}$) immediately gives the invariance of $\mathcal{L}$ under the transformations (1.3.32) and (1.3.33).

To familiarize ourselves with such a framework, let us see how to recast the electromagnetic theory discussed in the previous section within this formalism.

We see that, for this purpose, we need to specify the gauge group G, as given by the (abelian) group of rotations in $\mathbb{R}^2$, and the internal symmetry space, as given by the complex line. Namely, we take

$$G = U(1) \equiv SO(2) \text{ and } \mathbb{L} = \mathbb{C}, \tag{1.3.36}$$

and consider $\mathbb{C}$ equipped with the standard inner product

$$(z, w) = z\bar{w} \qquad \forall z, w \in \mathbb{C}. \tag{1.3.37}$$

We know that $U(1)$ defines a compact (topologically S^1) abelian Lie group, acting on $\mathbb{C}$ as multiplication by a unitary complex number. The associated Lie algebra $\mathcal{G}$ is the imaginary axis, which is represented as

$$\mathcal{G} = -i\mathbb{R}$$

and which acts on $\mathbb{C}$ by multiplication.

Consequently, for the matter and potential fields, we find that the expressions (1.2.2) and (1.2.3). Furthermore, in accordance with (1.3.27), (1.3.28) and (1.3.29) — the corresponding gauge (Maxwell) field and covariant derivative of ϕ — take the forms of (1.2.4), (1.2.5) and (1.2.6), respectively.

In addition, a gauge transformation g over $U(1)$ is assigned simply by a smooth function $\omega : \mathbb{R}^{1+d} \longrightarrow \mathbb{R}$ as follows:

$$\begin{aligned} g : \mathbb{R}^{1+d} &\longrightarrow U(1) \\ x &\longrightarrow e^{i\omega(x)}. \end{aligned}$$

With this information, we can revisit the Maxwell–Higgs theory discussed in the previous section to fit within the gauge field formalism illustrated above.

More general (non-abelian) gauge field theories may be formulated when we replace $U(1)$ with other (matrix) groups. Thus, while $U(1)$ pertains to electromagnetism, the group $SU(2)$ is involved in the formulation of the Yang–Mills–Higgs theory of weak interactions, while $SU(3)$ is the appropriate gauge group to describe strong interactions. In the group $SU(5)$ lies the hope for describing a universal unified theory beyond the already celebrated electroweak theory of Glashow–Salam–Weinberg which formulates a unified $(SU(2) \times U(1))$-gauge field theory for electromagnetic and weak interactions.

We shall present a more detailed discussion of non-abelian gauge field theories in the context of the (real or complex) adjoint representation, where both the potential and the matter fields are expressed on the same basis of the gauge algebra (real or complex). As is already apparent from the expression of the covariant derivative (1.3.30), it is important to consider on $\mathcal{G}$ (or $\mathcal{G}_{\mathbb{C}}$), a basis that satisfies the most convenient commutator relations.

We will reach this goal by means of the Cartan–Weyl basis decomposition (cf. [Ca], and [Hu]) which we discuss briefly in the following section; we will describe the most relevant features, especially in the context of semisimple Lie groups (e.g., $SU(n)$).

1.3.4 The Cartan–Weyl generators: basics

In this section we give a brief account on the Cartan–Weyl decomposition of a complex (finite-dimensional) semisimple Lie algebra $\{\mathbb{L}, [\ ,\ ]\}$ according to the (complex) adjoint representation; we refer to [Ca] and [Hu] for details. To this purpose, we recall that there exists an element $A \in \mathbb{L}$, such that the linear map

$$\begin{aligned} ad(A) : \mathbb{L} &\longrightarrow \mathbb{L} \\ X &\longrightarrow [A, X] \end{aligned} \tag{1.3.38}$$

admits only zero as a multiple degenerate eigenvalue, while the number of remaining non-zero eigenvalues is maximal. Such non-zero eigenvalues are called the roots of $\mathbb{L}$ and necessarily must be simple eigenvalues.

The eigenspace of the zero eigenvalue (i.e., Ker $ad(A)$) contains A, and by virtue of the *Jacobi identity*

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \tag{1.3.39}$$

defines a subalgebra of $\mathbb{L}$, which can be shown to be *abelian* (i.e., each pair of its elements commute under the Lie bracket operation).

Actually, the eigenspace coincides with the maximal abelian subalgebra of $\mathbb{L}$, and thus is independent of the choice of the element A. Such a maximal abelian subalgebra of $\mathbb{L}$ is known as the *Cartan subalgebra*, and its dimension r defines the *rank* of $\mathbb{L}$. Denoting by $\{H_a\}_{a=1,\dots,r}$ a basis for the Cartan subalgebra, we have:

$$[H_a, H_b] = 0 \quad \forall a, b = 1, \dots, r. \tag{1.3.40}$$

Since A belongs to the Cartan subalgebra we may write:

$$A = \gamma^a H_a. \tag{1.3.41}$$

Now, let λ be a root of $\mathbb{L}$ and denote by E_λ a corresponding eigenvector, called a *root operator*. Then

$$[A, E_\lambda] = \lambda E_\lambda. \tag{1.3.42}$$

By means of the Jacobi identity (1.3.39), we find that

$$\begin{aligned}[A, [H_a, E_\lambda]] &= -[H_a, [E_\lambda, A]] - [E_\lambda, [A, H_a]] \\ &= \lambda[H_a, E_\lambda] \quad \forall a = 1, \dots, r\end{aligned}$$

where we have used (1.3.40) and (1.3.41). That is, $[H_a, E_\lambda]$ is also an eigenvector corresponding to the (simple) eigenvalue λ, and consequently

$$[H_a, E_\lambda] = \alpha_a E_\lambda \quad \forall a = 1, \dots, r. \tag{1.3.43}$$

In this way, to each root λ we can associate a vector $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{C}^r$ known as a *root vector*. Note that a root vector is non-zero, i.e., $\alpha \neq 0$. It is possible to check that the root vector α is uniquely determined by λ, in the sense that different roots give rise to different root vectors. By (1.3.41) the following relation holds:

$$\lambda = \gamma^a \alpha_a. \tag{1.3.44}$$

Thus, if we denote by $\Delta \subset \mathbb{C}^r \setminus \{0\}$ the set of root vectors, we find a one-to-one correspondence between Δ and the set of roots of $\mathbb{L}$.

For this reason we can replace E_λ with E_α to identify the corresponding root operator. Furthermore $\{H_a, E_\alpha\}$ defines a basis for $\mathbb{L}$, known as the Cartan–Weyl basis, and for each $X \in \mathbb{L}$ we obtain the following *root space decomposition*:

$$X = \sum_{a=1}^{r} y^a H_a + \sum_{\alpha \in \Delta} y_\alpha E_\alpha. \tag{1.3.45}$$

Next notice that, if $\alpha = (\alpha_a)_{a=1,\dots,r}$ and $\beta = (\beta_a)_{a=1,\dots,r} \in \Delta$ are two root vectors of Δ, with $\lambda_\alpha = \gamma^a \alpha_a$ and $\lambda_\beta = \gamma^a \beta_a$ the corresponding roots, then by means of the Jacobi identity (1.3.39), we can verify the identity:

$$[A, [E_\alpha, E_\beta]] = (\lambda_\alpha + \lambda_\beta)[E_\alpha, E_\beta]. \tag{1.3.46}$$

Thus we see that, either E_α, E_β commute as

$$[E_\alpha, E_\beta] = 0, \tag{1.3.47}$$

or $\lambda_\alpha + \lambda_\beta$ is a root of $\mathbb{L}$ and we must distinguish between the following cases:

$$i)\ \lambda_\alpha + \lambda_\beta = 0, \text{ i.e., } \beta = -\alpha, \text{ where we find}$$
$$[E_\alpha, E_{-\alpha}] = \alpha^a H_a \text{ for suitable } \alpha^a \in \mathbb{C} \tag{1.3.48}$$
$$ii)\ \lambda_\alpha + \lambda_\beta \neq 0, \text{ i.e., } \alpha + \beta \text{ is a root of } A \text{ and we find}$$
$$[E_\alpha, E_\beta] = c_{\alpha\beta} E_{\alpha+\beta} \text{ for suitable } c_{\alpha\beta} \in \mathbb{C}. \tag{1.3.49}$$

As a matter of fact, when E_α and E_β commute, we can suppose that the relations (1.3.48) and (1.3.49) are still valid by taking $\alpha^a = 0$ or $c_{\alpha\beta} = 0$, respectively.

In particular, from identity (1.3.49) we see that in order to represent $\mathbb{L}$ we do not need to include $E_{\alpha+\beta}$, among its root operators when $\alpha + \beta \neq 0$; this eigenvector may be recovered via the bracket operation from E_α and E_β, provided they do not commute.

Therefore, it becomes an interesting problem to determine the minimal set Γ of root vectors, known as the *simple root vectors*, which generate the whole set Δ by summing over its elements. In this way, the Lie algebra $\mathbb{L}$ would be completely represented by the generators of the Cartan subalgebra and the simple root step operators.

The set of simple root step operators is particularly nice to describe for semisimple Lie algebra for which we can derive direct information out of the Killing form (1.3.11) in the root space decomposition.

To this purpose, let $\{X_l\}$ be a basis for $\mathbb{L}$ with structural constants $C^l_{m,n}$ defined by the relation:

$$[X_m, X_n] = C^l_{m,n} X_l. \tag{1.3.50}$$

Then, the Killing form k may be completely expressed in terms of the symmetric square matrix $g = g_{m,n}$, with

$$g_{m,n} = C^l_{m,\nu} C^\nu_{nl}, \tag{1.3.51}$$

as we find that

$$k(X, Y) = x^a g_{a,b} y^b, \quad \text{for } X = x^l X_l \quad \text{and } Y = y^l X_l.$$

Therefore, if $\mathbb{L}$ is semisimple, then $g = g_{m,n}$ is *invertible* (i.e., $\det g \neq 0$). Furthermore, if $\mathbb{L}$ corresponds to the complexification of a real Lie algebra relative to a compact, connected semisimple group, then by Theorem 1.3.1, we may even conclude that g is negative definite.

Let us see how to use this information when we fix $\{H_a, E_\alpha, a = 1, \ldots, r,\ \alpha \in \Delta\}$ as a basis for $\mathbb{L}$ according to the root space decomposition.

As a consequence of the commutator relations (1.3.40), (1.3.43), (1.3.48), and (1.3.49), we obtain the structural constants to be given as follows:

$$C_{ab}^{\beta} = 0 = C_{ab}^{l} \quad \beta \in \Delta,\ a, b, l \in \{1, \dots, r\} \tag{1.3.52}$$

$$C_{a\alpha}^{\beta} = \alpha_a \delta_{\alpha}^{\beta}, \quad C_{a\alpha}^{l} = 0 \quad \alpha, \beta \in \Delta,\ a, b, l \in \{1, \dots, r\} \tag{1.3.53}$$

$$C_{\alpha\mu}^{\beta} = \begin{cases} c_{\alpha\mu}\delta_{\alpha+\mu}^{\beta} & \text{if } \alpha + \mu \neq 0 \\ 0 \text{ otherwise} \end{cases} \qquad C_{\alpha\mu}^{l} = \begin{cases} 0 & \alpha + \mu \neq 0 \\ \alpha^a \delta_l^a & \alpha + \mu = 0 \end{cases} \tag{1.3.54}$$

$$\alpha, \beta, \mu \in \Delta,\ l \in \{1, \dots, r\}.$$

By virtue of (1.3.52), (1.3.53), and (1.3.54), we find that the matrix g is formed by two blocks: $\{g_{ab}\}_{a,b=1,\dots,r}$ given by its restriction over the Cartan subalgebra,

$$g_{ab} = k(H_a, H_b);$$

$\{g_{\alpha\beta}\}_{\alpha,\beta\in\Delta}$ given by its restriction over the root space operators

$$g_{\alpha\beta} = k(E_\alpha, E_\beta),$$

while g contains zeros elsewhere.

In fact, for $a \in \{1, \dots, r\}$ and $\alpha \in \Delta$ we find

$$g_{\alpha a} = C_{\alpha b}^{l} C_{al}^{b} + C_{\alpha\beta}^{l} C_{al}^{\beta} + C_{\alpha\beta}^{\nu} C_{a\nu}^{\beta} + C_{\alpha b}^{\nu} C_{a\nu}^{b} = 0 = g_{a\alpha}. \tag{1.3.55}$$

On the other hand, when $\alpha, \mu \in \Delta$ we have

$$g_{\alpha\mu} = C_{\alpha b}^{l} C_{\mu l}^{b} + C_{\alpha\beta}^{l} C_{\mu l}^{\beta} + C_{\alpha\beta}^{\nu} C_{\mu\nu}^{\beta} + C_{\alpha b}^{\nu} C_{\mu\nu}^{b} \left(= g_{\mu\alpha}\right).$$

We see that necessarily $-\alpha \in \Delta$; otherwise, we would get $g_{\alpha\mu} = 0$ for all $\mu \in \Delta$, and in light of (1.3.55), we would contradict the non-degeneracy of g.

Thus, by the analysis of the block $\{g_{\alpha\mu}\}$ we have discovered a first important property of semisimple Lie algebras; namely,

$$\text{if } \alpha \in \Delta,\ \text{then } -\alpha \in \Delta. \tag{1.3.56}$$

Next, we consider the block $\{g_{ab}\}_{a,b=1,\dots,r}$ relative to the restriction of g on the Cartan subalgebra.

First, note that $\{g_{ab}\}$ defines a non-degenerate $r \times r$ matrix. In view of (1.3.53) and (1.3.48), we compute

$$\begin{aligned} \alpha_a k(E_\alpha, E_{-\alpha}) &= k([H_a, E_\alpha], E_{-\alpha}) \\ &= k(H_a, [E_\alpha, E_{-\alpha}]) \\ &= \alpha^b k(H_a, H_b) \\ &= \alpha^b g_{ab}, \end{aligned} \tag{1.3.57}$$

where we have used (1.3.13).

Therefore, if we normalize the root operators to satisfy

$$k(E_\alpha, E_{-\alpha}) = 1,$$

then from (1.3.57) we deduce

$$\alpha^a = g^{ab}\alpha_b, \tag{1.3.58}$$

where, as usual, (g^{ab}) denotes the inverse matrix of (g_{ab}). By virtue of (1.3.58), it seems appropriate to introduce the following inner product over the set Δ:

$$(\alpha, \beta) = \alpha^a\beta_a = \beta_a g^{ab}\alpha_b. \tag{1.3.59}$$

Furthermore, in terms of the structural constants we see that

$$\begin{aligned} g_{ab} &= C^l_{am}C^m_{bl} + C^l_{a\alpha}C^\alpha_{bl} + C^\beta_{am}C^m_{b\beta} + C^\beta_{a\alpha}C^\alpha_{b\beta} \\ &= C^\beta_{a\alpha}C^\alpha_{b\beta} = \textstyle\sum_{\alpha\in\Delta}\alpha_a\alpha_b, \end{aligned}$$

and the non-degeneracy of $\{g_{ab}\}$ implies that the set of root vectors must span the whole space $\mathbb{C}^r$.

Such a property indicates that the set of simple roots Γ must contain exactly r independent vectors. This result can be made rigorous (cf. [Ca] and [Hu]) as it is possible to show that, for a semisimple Lie algebra the set of simple roots contains exactly r independent vectors and any other root vector can be obtained as a sum of simple root vectors with integer coefficients all having the same sign (either ≥ 0 or ≤ 0). More precisely,

$$\begin{aligned} &\Gamma = \left\{\alpha^{(a)}\ a = 1, \ldots, r\right\} \text{ and } \forall\alpha \in \Delta : \\ &\alpha = n_a\alpha^{(a)} \text{ with } n_a \in \mathbb{Z}^+\ \forall a,\ \text{ or } n_a \in \mathbb{Z}^-\ \forall a. \end{aligned} \tag{1.3.60}$$

The root operator corresponding to the simple root $\alpha^{(a)}$ will be denoted by E_a, $a = 1, \ldots, r$ and called a *simple root step operator*. Note that, according to (1.3.60), for $a \neq b$ we have

$$[E_a, E_{-b}] = 0,$$

as $\alpha^{(a)} - \alpha^{(b)} \notin \Delta$.

In fact, according to Chevalley's normalization conditions (cf. [Ca], [Hu], and [Chev]), it is always possible to arrange the Cartan subalgebra generators $\{H_a\}_{a=1,\ldots,r}$ and the simple root step operators $\{E_a\}_{a=1,\ldots,r}$ to satisfy the following commutator and trace relations:

$$[H_a, H_b] = 0 \tag{1.3.61}$$

$$[E_a, E_{-b}] = \delta^a_b H_a \tag{1.3.62}$$

$$[H_a, E_{\pm b}] = \pm K_{ab}E_{\pm b} \tag{1.3.63}$$

$$\text{tr}\,(E_a E_{-b}) = \delta^a_b,\quad \text{tr}\,(H_a H_b) = K_{ab},\quad \text{tr}\,(H_a E_{\pm b}) = 0 \tag{1.3.64}$$

$$a, b = 1, \ldots, r,$$

where the coefficients

$$K_{ab} = 2\frac{\left(\alpha^{(a)}, \alpha^{(b)}\right)}{\left(\alpha^{(a)}, \alpha^{(a)}\right)} \tag{1.3.65}$$

form an $r \times r$ matrix, known as the *Cartan matrix*.

The above decomposition can be nicely illustrated for the group $SU(n)$, where the corresponding operators H_a and E_a are determined explicitly. We start with the particularly simple case of $SU(2)$, where the associated Lie algebra

$$su(2) = \left\{A \in gl(2, \mathbb{C}) : \ A^\dagger = -A, \ \text{tr}\ A = 0\right\}$$

admits (real) dimension 3. Its complexification

$$sl(2, \mathbb{C}) = \{A \in gl(2, \mathbb{C}) : \ \ \text{tr}\ A = 0\}$$

also admits (complex) dimension 3.

We easily determine a basis for $sl(2, \mathbb{C})$, as given by:

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Observe that $T_3 = T_2^\dagger$, and the following commutator relations hold:

$$[T_1, T_2] = 2T_2, \qquad [T_1, T_3] = -2T_3, \tag{1.3.66}$$

$$[T_2, T_3] = T_1. \tag{1.3.67}$$

Hence, no pair of linearly independent elements of $sl(2, \mathbb{C})$ can commute. Therefore, the corresponding Cartan subalgebra is one-dimensional (i.e., $r = 1$), and in view of (1.3.66) we arrive at the conclusion that

$$H_1 = T_1, \ E_1 = T_2$$

give the Cartan subalgebra generator and the simple root step operator, respectively, while $E_{-1} = E_1^\dagger = T_3$. It can be checked by direct inspection that all the Chevallery's relations (1.3.61)–(1.3.64) are verified since $K = K_{11} = 2$ in this case.

To obtain a basis for $su(2)$, we simply have to use linear (complex) combinations of H_1, E_1 and E_{-1} to obtain three linearly independent anti-Hermitian traceless matrices. Easily we guess them to be:

$$iH_1, \ E_1 - E_{-1}, \ i(E_1 + E_{-1}).$$

In fact, it is more convenient to take as a basis for $su(2)$ the following:

$$X_1 = \frac{1}{2i}(E_1 + E_{-1}), \ X_2 = -\frac{1}{2}(E_1 - E_{-1}), \ X_3 = \frac{1}{2i}H_1. \tag{1.3.68}$$

This basis satisfies

$$[X_a, X_b] = \varepsilon_{abc} X_c, \tag{1.3.69}$$

and is related to the (Hermitian) Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E_1 + E_{-1}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i(E_{-1} - E_1), \tag{1.3.70}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H_1 \tag{1.3.71}$$

via the identity $X_a = \frac{1}{2i}\sigma_a$, $a = 1, 2, 3$.

This first example gives a hint on how to proceed more generally for the group $SU(n)$. We introduce the matrices T_{ab} defined by the condition:

$$(T_{ab})_{jk} = \delta^a_j \delta^b_k \tag{1.3.72}$$

$a, b = 1, \ldots, n$. Set

$$H_a = T_{aa} - T_{a+1,a+1}, \ a = 1, \ldots, n-1, \tag{1.3.73}$$

and note that such matrices are diagonal and hence commute as

$$[H_a, H_b] = 0, \ a, b = 1, \ldots, n-1.$$

In addition, $T^\dagger_{ab} = T_{ba}$ and $T_{ab} \in sl(n, \mathbb{C})$ only if $a \neq b$. In fact, the set

$$\left\{H_a, T_{ab}, T^\dagger_{ab}, \ 1 \leq a < b \leq n\right\}$$

forms a basis for $sl(n, \mathbb{C})$.

The commutator relations for the matrice T_{ab} can be easily computed to yield the following relations:

$$[T_{ab}, T_{cd}] = \begin{cases} T_{aa} - T_{cc} \ \text{ if } a = d, \ b = c \\ T_{ad} \ \text{ if } a \neq d, \ b = c \\ -T_{cb} \ \text{ if } a = d, \ b \neq c \\ 0 \ \text{ if } a \neq d, \ b \neq c. \end{cases} \tag{1.3.74}$$

As a consequence of (1.3.73) and (1.3.74), we also establish that

$$[H_a, T_{bc}] = K^a_{bc} T_{bc}, \tag{1.3.75}$$

where for $b < c$, we have

$$K^a_{bc} = \begin{cases} 0 \ \text{ if } b, c \notin \{a, a+1\} \\ 1 \ \text{ if } a = b, \ \text{ and } c \neq a+1 \\ 2 \ \text{ if } a = b, \ \text{ and } c = a+1 \\ -1 \ \text{ if } b = a+1 \text{ or } c = a \end{cases} \tag{1.3.76}$$

and $K^a_{cb} = -K^a_{bc}$, $a, b, c \in \{1, \ldots, n\}$.

These relations imply that the Cartan subalgebra of $sl(n, \mathbb{C})$ is generated by the (linearly independent) matrices H_a $a = 1, \dots, n-1$; and thus we find $r = n - 1$ for the corresponding rank. Furthermore, over the elements of the Cartan subalgebra, each matrix, T_{ab} $a \neq b$, acts as a root operator. In addition, observe that the set of linearly independent elements

$$\{T_{a,a+1},\ a = 1, \dots, n-1\}$$

generate the whole Lie algebra $sl(n, \mathbb{C})$ via the commutator and the Hermitian conjugation operations. Therefore,

$$H_a = T_{aa} - T_{a+1,a+1} \text{ and } E_a = T_{a,a+1}\ a = 1, \dots, n-1, \tag{1.3.77}$$

must correspond, respectively, to the Cartan subalgebra generators and the simple root step operators, with

$$E_{-a} = E_a^\dagger = T_{a+1,a},\ a = 1, \dots, n-1. \tag{1.3.78}$$

Consequently the $(n-1) \times (n-1)$ Cartan (non-singular) matrix K is given as

$$K = \begin{bmatrix} 2 & -1\ 0 & \dots\ \dots\ 0 \\ -1\ 2 & -1\ 0 & \dots\ 0 \\ \dots\ \dots\ \dots & \dots\ \dots & \\ \dots\ \dots\ \dots & \dots\ \dots & \\ 0 & \dots\ \dots\ -1\ 2 & -1 \\ 0 & \dots\ \dots\ \dots & -1\ 2 \end{bmatrix}, \tag{1.3.79}$$

and $\det K = n$.

By direct inspection, it is not difficult to check that the Chevalley normalization condition, (1.3.61)–(1.3.64), is valid for the given Cartan–Weyl generators (1.3.77) and (1.3.78).

Exactly as above, we may now determine a basis for the (real) Lie algebra $su(n)$ as given by the following $n^2 - 1$ elements:

$$\begin{aligned} &\{iH_a,\ E_a - E_{-a},\ i(E_a + E_{-a}),\ [E_b, E_c] + [E_{-b}, E_{-c}], \\ &i([E_b, E_c] - [E_{-b}, E_{-c}]);\ a = 1, \dots, n-1,\ 1 \leq b < c \leq n-1\}. \end{aligned} \tag{1.3.80}$$

1.3.5 Yang–Mills–Higgs theory

The Yang–Mills–Higgs theory (YMH-theory) has been introduced in physics literature to describe weak particle interaction. In this respect, the appropriate gauge group to consider is $SU(2)$ in the (real) adjoint representation.

More generally, we are going to formulate the theory over a gauge group G, assumed compact, connected and semisimple, so that $SU(n)$ (and thus $SU(2)$) is included as a particular case.

In the context of the (real) adjoint representation, both the potential $\mathcal{A}$ and the matter field ϕ take values over the gauge algebra $\{\mathcal{G}, [\cdot, \cdot]\}$.

Notice also that in this situation it is possible to define an invariant inner product $(\cdot,\cdot)$ over $\mathcal{G}$, which commutes nicely under the Lie bracket operation. Namely, the product satisfies

$$(Ad(g)A, Ad(g)B) = (A, B)\ \forall g \in G,\ \forall A, B \in \mathcal{G} \tag{1.3.81}$$

and

$$([A, B], C) = (A, [B, C])\ \forall A, B, C \in \mathcal{G}. \tag{1.3.82}$$

For example, in view of (1.3.12), (1.3.13), and Theorem 1.3.1, we may take the inner product as the Killing form with the opposite sign.

For the case $G = SU(n)$, it is easy to check that the usual inner product over $gl(n, \mathbb{C})$ is given by

$$(A, B) = \text{ tr } AB^{\dagger} \tag{1.3.83}$$

and satisfies (1.3.81) and (1.3.82) when restricted over the elements of $su(n)$. In fact, on $su(n)$ we find that $(A, B) = -\text{tr}AB$, and it is well-known that the Killing form and the *trace* form, $(A, B) \longrightarrow \text{tr}AB$, are multiples of each another.

In terms of the (smooth) $\mathcal{G}$-valued components of the potential field A_α $(\alpha = 0, 1, \ldots, d)$ and of the matter field ϕ, the Yang–Mills–Higgs Lagrangean density takes the expression

$$\mathcal{L}_{YMH}(\mathcal{A}, \phi) = -\frac{1}{4}\left(F_{\alpha\beta}, F^{\alpha\beta}\right) + \frac{1}{2}\left(D_\alpha\phi, D^\alpha\phi\right) - \frac{\lambda}{8}\left(|\phi|^2 - 1\right)^2, \tag{1.3.84}$$

with $|\phi|^2 = (\phi, \phi),\ \lambda \in \mathbb{R}^+$, and

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + \left[A_\alpha, A_\beta\right];\ D_\alpha\phi = \partial_\alpha\phi + [A_\alpha, \phi]. \tag{1.3.85}$$

By virtue of (1.3.82), we derive the corresponding *Yang–Mills–Higgs equations* of motion as

$$\begin{cases} D_\alpha D^\alpha\phi = \frac{\lambda}{2}\phi\left(1 - |\phi|^2\right), \\ D_\beta F^{\alpha\beta} = [\phi, D^\alpha\phi] \end{cases} \tag{1.3.86}$$

(cf. [JT]).

To obtain a selfdual reduction principle, first we focus on the "pure" Yang–Mills Lagrangean density,

$$\mathcal{L}_{YM} = -\frac{1}{4}\left(F_{\alpha\beta}, F^{\alpha\beta}\right), \tag{1.3.87}$$

where we neglect both the Higgs field and the scalar potential by taking $\phi = 0$ and $\lambda = 0$ in (1.3.84). The corresponding *Yang–Mills equations*

$$D_\beta F^{\alpha\beta} = 0, \tag{1.3.88}$$

may be considered as the non-abelian counterpart of Maxwell's equations in the vacuum (1.2.18). Therefore, if we are restricted to *static* finite action solutions for (1.3.88) (known as Yang–Mills fields), then in the temporal gauge (which allows us to take $A_0 = 0$) for spatial dimension $d = 4$, the Bianchi identity relative to the curvature $F_{\mathcal{A}}$,

$$D_\gamma F_{\mu\nu} + D_\mu F_{\nu\gamma} + D_\nu F_{\gamma\mu} = 0, \tag{1.3.89}$$

may be rewritten in the form

$$D_l \left(\frac{1}{2} \varepsilon_{jlkn} F^{kn} \right) = 0, \tag{1.3.90}$$

for $j = 1, \dots, 4$. (Compare (1.3.90) with (1.2.8).) Identity (1.3.90) calls our attention to the $*$ Hodge operator, which transforms a p-form over $\mathbb{R}^d$

$$\omega = \omega_{i_1,\dots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

into the (d−p)-form

$$*\omega = \frac{1}{p!} \varepsilon^{i_1,\dots,i_p,j_1,\dots,j_{d-p}} \omega_{i_1,\dots,i_p} dx^{j_1} \wedge \cdots \wedge dx^{j_{d-p}}.$$

In fact, in terms of the 2-form $F_{\mathcal{A}}$ in (1.3.27), equation (1.3.90) simply states that

$$D_{\mathcal{A}}(*F_{\mathcal{A}}) = 0. \tag{1.3.91}$$

Consequently, we deduce that every connection $\mathcal{A}$ satisfying the selfdual/antiselfdual relation,

$$F_{\mathcal{A}} = \pm * F_{\mathcal{A}}, \tag{1.3.92}$$

automatically solves (1.3.88).

In other words, we may regard (1.3.92) as a first-order factorization of the (second-order) Yang–Mills equations (1.3.88). Their finite action solutions, known as the selfdual/antiselfdual *instantons*, have been studied in great detail, especially for what concerns the gauge group $G = SU(n)$. Indeed, due to Uhlenbeck's removable singularity result (cf. [U1] and [U2]), instantons over $\mathbb{R}^4$ can be extended at infinity to define (in a suitable gauge) a smooth connection over S^4. In this way one finds that $SU(n)$-instantons correspond to minima of the Yang–Mills energy $\mathcal{E}_{YM} (= \mathcal{L}_{YM})$ when restricted to those connections over S^4 with fixed second Chern number $N \in \mathbb{Z}$. The corresponding minimal energy is easily computed and given as,

$$\mathcal{E}_{YM} = 8\pi^2 |N|, \tag{1.3.93}$$

where the sign of N simply allows one to decide whether to consider the selfdual or the antiselfdual solutions of (1.3.92).

For each $N \in \mathbb{Z}$, explicit expressions of such (minimal) N-instantons have been obtained in terms of the basis (1.3.80), see e.g., [BPST], [ADHM], [AHS2], [JNR], [JR], ['tH1] and [Wit1].

The dimension of the manifold of N-instantons (the space of moduli) has been computed (in terms of N) by [AHS1], [Schw] and [JR]. Thus, in analogy to the de Rahm cohomology of an oriented compact manifold, where each class may be represented by a harmonic form (cf. [Jo]), we have that each second Chern–Pontryagin class of S^4 may be represented by a family (i.e., the moduli space) of $SU(2)$-instantons over S^4.

See [Y1] for a possible extension of the above analysis to the dimension $d = 4m$, $m \in \mathbb{N}$.

It should be mentioned, however, that selfdual/antiselfdual instantons do not exhaust the whole class of Yang–Mills fields with finite energy as shown in [SSU], [Par], [SS], [Bor], [Bu1], [Bu2], and [Ta3]. On the other hand, there are classes of Yang–Mills fields that can be recognized as instantons, for example, the local minima for $\mathcal{L}_{YM}$ (cf. [BoL]) or the $O(3)$-symmetric solutions of (1.3.88). These all share features with the vortex configurations of the abelian Maxwell–Higgs model (1.2.9) (see [JT]).

In this respect, recall that any finite energy static solution of the abelian Maxwell–Higgs equations (1.2.13)–(1.2.15) in $\mathbb{R}^2$ corresponds to a Nielsen–Olesen vortex (cf. [NO]), in the sense that it actually satisfies the selfdual equations (1.2.25)–(1.2.27) (cf. [Ta2] and [JT]). Moreover, such vortices can be classified by topological classes still labelled by an integer N. This integer now corresponds to the degree of the Higgs field, and thus represent a homotopy class of the abelian gauge group $U(1)$. In other words,

$$N \in \mathbb{Z} = \pi_1\Big(U(1)\Big).$$

Thus, the above claim about the Chern–Pontryagin classes now states that each homotopy class of $\pi_1\Big(U(1)\Big)$ can be represented by a family of Nielsen–Olesen vortices, solutions of (1.2.26)–(1.2.27) (cf. [JT]). The moduli space formed by such N-vortices has been completely characterized in [Ta1] as a manifold equivalent to $\mathbb{R}^{2N}$ modulo the group of permutation of N elements (see also [JT], [Bra1], [Bra2], [Ga1], [Ga2], [Ga3], and [WY]).

We shall see that each element of $\pi_1\Big(U(1)\Big)$ is also represented by a family of (abelian) selfdual Chern–Simons vortices. However, the description of the moduli space formed by Chern–Simons N-vortices is far from complete in this case.

Returning to the selfduality property of the Yang–Mills–Higgs theory, (1.3.84), we mention the following dimensional reduction procedure that yields to selfduality for (1.3.84) in dimension $d = 3$ and for $\lambda = 0$ (i.e., no scalar potential is included in the Lagrangean $\mathcal{L}_{YMH}$).

More precisely still, in the static case and in the temporal gauge (according to which we can always take $A_0 = 0$), we consider the selfdual/antiselfdual equation (1.3.92) when the component A_j of the potential is independent of the x^4-variable, for every $j = 1, 2, 3, 4$, and set, $A_4 = \phi$. Then we easily check that they reduce to the equation:

$$*F_{\mathcal{A}} = \pm D_{\mathcal{A}}\phi, \tag{1.3.94}$$

which furnishes a first-order selfdual reduction of the second-order equations (1.3.86) when $d = 3$ and $\lambda = 0$.

The finite action solutions of (1.3.94) define the well-known *monopoles*, which have attracted much attention by virtue of their physical and mathematical properties, e.g., see [Le], [AtH], [Pe], [GO], and [PS].

They share with instantons (where $d = 4$, $\lambda = 0$ and $\phi = 0$) and vortices (where $d = 2$ and $\lambda = 1$) many elements of interest as shown in [JT].

1.3.6 A selfdual non-abelian Chern–Simons model

The goal of this section is to explore selfduality when the Yang–Mills Lagrangean (1.3.87) is replaced by the (non-abelian) Chern–Simons Lagrangean of (1.3.84).

Again this plan works through a dimensional reduction procedure (cf. [Jo] and [D1]). In the abelian case, this is evident when we compare (1.3.92) with (1.2.30). In the non-abelian case, we refer to a non-relativistic model discussed in detail in [D1]. This model's selfdual equations can be recast in terms of (1.3.92) by performing an appropriate transformation (in the spirit shown above) from four to two dimensions.

We start by noting that the YMH-theory, described in the previous section, would furnish a more direct non-abelian version of the abelian Maxwell–Higgs theory (1.2.9) when we express the gauge group according to the complex adjoint (i.e., conjugate) representation.

To be more precise and also to avoid useless technicalities, we shall focus on the gauge group $G = SU(n)$, where we have available the invariant inner product (1.3.83) on $sl(n, \mathbb{C})$ (and thus on $su(n)$) to carry out the explicit calculations.

It will be clear from the following discussion how to treat more general compact, connected semisimple gauge groups.

In the conjugate representation of $SU(n)$, the potential field component A_α is given by a smooth $su(n)$-valued map for $\alpha = 0, 1, \ldots, d$. In contrast, the Higgs field ϕ is a smooth $sl(n, \mathbb{C})$-valued map. Thus, we can still use the relations (1.3.85) to express the corresponding Lagrangean density as

$$\mathcal{L}(\mathcal{A}, \phi) = -\frac{1}{4} \operatorname{tr}\left(F_{\alpha\beta}\left(F^{\alpha\beta}\right)^{\dagger}\right) + \frac{1}{2} \operatorname{tr}\left(D_\alpha\phi\left(D^\alpha\phi\right)^{\dagger}\right) - V, \tag{1.3.95}$$

where $V = V(\phi, \phi^\dagger)$ is the gauge-invariant scalar potential.

Recall that for the scalar product (1.3.83), property (1.3.82) remains valid only over triplets A, B, C such that $B \in su(n)$ and $A, C \in sl(n, \mathbb{C})$. Then from (1.3.95), we deduce the following Euler–Lagrange equations:

$$D_\alpha D^\alpha \phi = -2\frac{\partial V}{\partial \phi^\dagger}, \tag{1.3.96}$$

$$D_\beta F^{\alpha\beta} = -i J^\alpha, \tag{1.3.97}$$

with

$$J^\alpha = \frac{i}{2}\left(\left[D^\alpha\phi, \phi^\dagger\right] - \left[\phi, \left(D^\alpha\phi\right)^\dagger\right]\right). \tag{1.3.98}$$

Here J^α, the corresponding (relativistic) non-abelian current, is covariantly conserved ($D_\alpha J^\alpha = 0$), in perfect analogy with the abelian counterpart (1.2.15).

We investigate selfduality, when the Yang–Mills term in the Lagrangean density (1.3.95) is replaced by the (lower-order) Chern–Simons term.

Therefore, we turn our attention to a planar (i.e., $d = 2$) Chern–Simons–Higgs theory described by the Lagrangean density

$$\mathcal{L}(\mathcal{A}, \phi) = k\mathcal{L}_{cs}(\mathcal{A}) + \operatorname{tr}\left(D_\alpha\phi\left(D^\alpha\phi\right)^\dagger\right) - V \tag{1.3.99}$$

with the non-abelian Chern–Simons Lagrangean

$$\mathcal{L}_{cs}(\mathcal{A}) = \frac{\varepsilon^{\mu\nu\alpha}}{2}\operatorname{tr}\left(A_\mu\partial_\nu A_\alpha + \frac{2}{3}A_\mu A_\nu A_\alpha\right) \qquad \alpha, \mu, \nu = 0, 1, 2. \tag{1.3.100}$$

Notice that in the abelian case, the second term in (1.3.100) drops out and we recover (1.2.29). In (1.3.99), the parameter $k > 0$ defines the Chern–Simons coupling parameter whose strength must be counter balanced by the strength of the scalar potential $V = V(\phi, \phi^\dagger)$, in order for (1.3.99) to support a selfdual structure.

To this purpose, Dunne in [D1] (see also [D2]) proposed the following gauge-invariant scalar potential:

$$\begin{aligned} V &= \frac{1}{k^2}\operatorname{tr}\left(\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - \nu^2\phi\right)\left(\left[\left[\phi, \phi^\dagger\right]\phi\right] - \nu^2\phi\right)^\dagger\right) \\ &= \frac{1}{k^2}\left|\left[\left[\phi, \phi^\dagger\right], \phi\right] - \nu^2\phi\right|^2, \end{aligned} \tag{1.3.101}$$

with the symmetry-breaking mass-scale parameter ν^2.

Although the potential V in (1.3.101) might appear unusual at first sight, we uncover its familiar nature by looking at it over the subspace of $sl(n, \mathbb{C})$, generated by the simple root step operators $E_a,\ a = 1, \ldots, n-1$ in (1.3.77). In fact, for $\phi = \phi^a E_a$, with $\phi^a \in \mathbb{C}$, we find

$$V = \frac{1}{k^2}|\phi^a(\nu^2 - K_{ba}|\phi^b|^2)|^2, \tag{1.3.102}$$

which might be taken as the natural extension over the $(n-1)$-ples $\{\phi^a\}$ of the selfdual Chern–Simons potential (1.2.40) with coupling provided by the Cartan matrix in (1.3.79).

The Euler–Lagrange equations relative to (1.3.95) are given by

$$D_\alpha D^\alpha\phi = -\frac{\partial V}{\partial\phi^\dagger}, \tag{1.3.103}$$

$$\frac{k}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta} = -iJ^\mu, \tag{1.3.104}$$

with

$$J^\mu = i\left(\left[D^\mu\phi, \phi^\dagger\right] - \left[\phi, (D^\mu\phi)^\dagger\right]\right), \tag{1.3.105}$$

as the corresponding non-abelian covariantly conserved current. Again we notice an obvious formal analogy between (1.3.105) and its abelian counterpart (1.2.35).

From the $\mu = 0$ component of (1.3.104) we obtain the (non-abelian) Gauss law constraint:

$$F_{12} = -\frac{1}{k}([\phi^\dagger, D^0\phi] - [(D^0\phi)^\dagger, \phi]). \tag{1.3.106}$$

As for the abelian case, we can use (1.3.106) to deduce the following expression for the gauge-invariant part of the energy density:

$$\mathcal{E} = \text{ tr}\left(D_\alpha\phi(D^\alpha\phi)^\dagger\right) + V \tag{1.3.107}$$

where again, the Chern–Simons term does not contribute to (1.3.107).

To reveal how the choice of V in (1.3.101) implies a selfdual structure, we record the following identity as the non-abelian counterpart of (1.2.38):

$$|D_1\phi|^2 + |D_2\phi|^2 = |D_\pm\phi|^2 \pm i \text{ tr}F_{12}\left[\phi, \phi^\dagger\right] \pm \frac{\varepsilon^{jk}}{2}\partial_j Q_k. \tag{1.3.108}$$

Here

$$D_\pm\phi = D_1\phi \pm i D_2\phi; \tag{1.3.109}$$

and

$$Q^\mu = i \text{ tr}\left(\phi^\dagger D^\mu - \left(D^\mu\phi\right)^\dagger \phi\right) \tag{1.3.110}$$

identifies the *abelian* current relative to the $U(1)$-invariance of the theory.

With this information and with the choice of the scalar potential V in (1.3.101), we can proceed analogously to the abelian situation and arrive at the following expression for the energy density:

$$\begin{aligned}
\mathcal{E} &= |D_0\phi|^2 + |D_\pm\phi|^2 \pm i \text{ tr}F_{12}\left[\phi, \phi^\dagger\right] + \frac{1}{k^2}\left|\left[\left[\phi, \phi^\dagger\right], \phi\right] - \nu^2\phi\right|^2 \pm \frac{\varepsilon^{jk}}{2}\partial_j Q_k \\
&= \left|D_0\phi \mp \frac{i}{k}\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - \nu^2\phi\right)\right|^2 + |D_\pm\phi|^2 \\
&\quad \pm i\frac{\nu^2}{k} \text{ tr}\left(D_0\phi\phi^\dagger - \phi\left(D_0\phi\right)^\dagger\right) \pm \frac{\varepsilon^{jk}}{2}\partial_j Q_k.
\end{aligned} \tag{1.3.111}$$

Thus, by using the elementary properties of traces and the component Q^0 of the abelian current (1.3.110), we find

$$\mathcal{E} = \left|D_0\phi \mp \frac{i}{k}\left(\left[\left[\phi, \phi^\dagger\right], \phi\right] - \nu^2\phi\right)\right|^2 + |D_\pm\phi|^2 \pm \frac{\nu^2}{k}Q^0 \pm \frac{\varepsilon^{jk}}{2}\partial_j Q_k. \tag{1.3.112}$$

Therefore, within appropriate boundary conditions that allow us to neglect the total spatial divergence term in (1.3.112), we can saturate the energy lower bound through the solutions of the following (first-order) equations:

$$D_{\pm}\phi = 0 \tag{1.3.113}$$
$$D_0\phi = \pm\tfrac{i}{k}\left(\left[\left[\phi,\phi^{\dagger}\right],\phi\right] - \nu^2\phi\right), \tag{1.3.114}$$

to be satisfied in addition to the Gauss law constraint (1.3.106). In other words, (1.3.106), (1.3.113), and (1.3.114) represent the *selfdual equations* corresponding to the non-abelian (pure) Chern–Simons model (1.3.99) and (1.3.101). In order to identify the *static* selfdual solutions (i.e., solutions independent of the x^0-variable), it is convenient to substitute (1.3.114) into (1.3.106) and rewrite the (static) selfdual equations in the form

$$\begin{aligned} D_{\pm}\phi &= 0 \\ F_{12} &= \mp\frac{2i}{k^2}\left[\nu^2\phi - \left[\left[\phi,\phi^{\dagger}\right],\phi\right],\phi^{\dagger}\right]; \end{aligned} \tag{1.3.115}$$

and then use (1.3.114) to determine only the A_0-component of the potential field.

Incidentally, notice that the selfdual equations (1.3.115) are trivialized in the real adjoint represenation where $\phi \in su(n)$, and so $\phi^{\dagger} = -\phi$. Indeed in this case, the second equation in (1.3.115) leads to $F_{12} = 0$, while (1.3.114) can only result in the trivial solution $\phi = 0$ since its left-hand side is Hermitian while its right-hand side is anti-Hermitian.

So the selfdual Chern–Simons theory presented here is of true interest in the conjugate representation. The resulting (non-abelian) selfdual equations (1.3.114) and (1.3.115) are much more difficult to handle in comparison with their abelian counterpart (1.2.45), in spite of their analogies at a “formal” level.

In fact, concerning (1.3.114) and (1.3.115) rigorous analytical results have been obtained only under the ansatz that each component A_α of the potential field takes values on the Cartan subalgebra of $su(n)$, while the matter field ϕ belongs to the subspace of $sl(n,\mathbb{C})$ generated by the simple root step operators. Observe that the commutator relations (1.3.61)–(1.3.64), valid for the Cartan–Weyl generators (1.3.77), prove consistency of the selfdual equations under such restrictions. More precisely, we let

$$A_\alpha = -iA^a_\alpha H_a \quad A^a_\alpha \in \mathbb{R},\ \alpha = 0,1,2 \tag{1.3.116}$$
$$\phi = \phi^a E_a \quad \phi^a \in \mathbb{C}, \tag{1.3.117}$$

$a = 1,\ldots,n-1$; and see that

$$D_\alpha\phi = \left(D^a_\alpha\phi^a\right)E_a \text{ with } D^a_\alpha\phi^a = \partial_\alpha\phi^a - i\left(A^b_\alpha K_{ba}\right)\phi^a, \tag{1.3.118}$$

where K_{ba} are the coefficients of the Cartan matrix. Furthermore,

$$F_{\alpha\beta} = -iF^a_{\alpha\beta}H_a \text{ with } F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha, \tag{1.3.119}$$

where no commutator appears, as $\{H_a, a = 1,\ldots,n-1\}$ commute; and

$$J_0 = i\left(\left[D_0\phi,\phi^{\dagger}\right] - \left[\phi,(D_0\phi)^{\dagger}\right]\right) = J^a_0 H_a, \tag{1.3.120}$$

with

$$J_0^a = i\left(\overline{\phi^a} D_0^a \phi^a - \overline{D_0^a \phi^a}\phi^a\right). \tag{1.3.121}$$

Therefore, setting

$$D_\pm^a \phi^a = D_1^a \phi^a \pm i D_2^a \phi^a, \tag{1.3.122}$$

we can express, by straighforward calculations, the selfdual equations (1.3.115) componentwise as

$$\begin{aligned} D_\pm^a \phi^a &= 0, \\ F_{12}^a &= \pm\frac{2}{k^2}\left(\nu^2 - |\phi^b|^2 K_{ba}\right)|\phi^a|^2; \end{aligned} \tag{1.3.123}$$

while the Gauss law constraint (1.3.106) becomes

$$F_{12}^a = \frac{1}{k} J_0^a, \tag{1.3.124}$$

$a = 1, \ldots, n-1$.

Also notice that, for the abelian current density, we have

$$Q^0 = \sum_{a=1}^{n-1} J_0^a. \tag{1.3.125}$$

In other words, a solution to (1.3.123) and (1.3.124) give rise (via (1.3.116) and (1.3.117)) to the solutions for the selfdual equations (1.3.114) and (1.3.115), whose energy density reduces to

$$\mathcal{E} = \pm\nu^2 \sum_{a=1}^{n-1} F_{12}^a + \text{ spatial divergence terms.} \tag{1.3.126}$$

It is not surprising that for $n = 2$, the reduced SU(2)-selfdual equations above just coincide with the selfdual equations of the abelian U(1)-Chern–Simons model (1.2.41).

On the other hand, for $n \geq 3$, the selfdual equations (1.3.123) and (1.3.124) represent a system of selfdual abelian-type equations coupled through the Cartan matrix (1.3.79). This fact is also expressed by the nature of the gauge invariance properties of equations (1.3.123) and (1.3.124), given by the transformation rules:

$$A_\alpha^a \longrightarrow A_\alpha^a + \partial_\alpha \omega_a, \tag{1.3.127}$$

$$\phi^a \longrightarrow e^{i\omega^b K_{ba}} \phi^a. \tag{1.3.128}$$

We refer to [D1] and [Y1] for a discussion of other non-abelian Chern–Simons models concerning non-relativistic settings.

1.4 Selfduality in the electroweak theory

As a last example, we present Ambjorn Olesen's approach (cf. [AO1], [AO2], and [AO3]) to describe a selfdual structure for the celebrated electroweak theory of Glashow–Salam–Weinberg (cf. [La]) of unified electromagnetic and weak forces.

The electroweak theory is a relativistic $(SU(2) \times U(1))$-gauge field theory that we are going to consider over the Minkowski space $(\mathbb{R}^{1+3}, g)$ with metric tensor $g = \mathrm{diag}(1, -1, -1, -1)$. The gauge group $G = SU(2) \times U(1)$ acts over $\mathbb{C}^2$ (the internal symmetry space) according to the representation

$$\rho : SU(2) \times U(1) \longrightarrow \mathrm{Aut}(\mathbb{C}^2). \tag{1.4.1}$$

We are going to describe this representation in terms of the matrices:

$$t_a = \frac{1}{2}\sigma_a, \; a = 1, 2, 3, \tag{1.4.2}$$

where σ_a is the 2×2 Pauli matrix and $a = 1, 2, 3$, is defined in (1.3.70) and (1.3.71). To this purpose, note that by virtue of (1.3.68) and (1.3.69), we know that $\{-it_a\}_{a=1,2,3}$ defines a basis for $su(2)$, and there holds:

$$[t_a, t_b] = i\varepsilon_{abc}t_c, \tag{1.4.3}$$

$$\mathrm{Tr}\; t_a t_b = \delta^a_b, \tag{1.4.4}$$

where $\mathrm{Tr} = 2\mathrm{tr}$ is the normalized trace.

Hence, we can use the exponential map (1.3.24) to express every element of $SU(2)$ as follows:

$$e^{-i\lambda^a t_a} \in SU(2), \; \lambda^a \in \mathbb{R}. \tag{1.4.5}$$

Moreover, letting

$$t_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.4.6}$$

we may extend the notation in (1.4.5) over the group $U(1)$, by representing every element of $U(1)$ as a 2×2 complex matrix in the form:

$$e^{-i\xi t_0} \in U(1), \; \xi \in \mathbb{R}. \tag{1.4.7}$$

Then, for $h = (e^{-i\lambda^a t_a}, e^{-i\xi t_0}) \in G = SU(2) \times U(1)$ and $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathbb{C}^2$, the representation ρ of G on $\mathbb{C}^2$ is defined as

$$\rho(h)\varphi = e^{-ig\lambda^a t_a - ig_*\xi t_0}\varphi \in \mathbb{C}^2, \tag{1.4.8}$$

with g and g_* the coupling constants relative to the group $SU(2)$ and $U(1)$, respectively.

In this setting, the *potential field* is expressed by the pair

$$\begin{aligned} \mathcal{A} &= -igA_\alpha dx^\alpha \ \text{ with } A_\alpha = A_\alpha^a t_a, \\ B &= -ig_* B_\alpha t_0 dx^\alpha, \end{aligned} \tag{1.4.9}$$

where A_α^a, $a = 1, 2, 3$ and B_α are smooth real functions over $\mathbb{R}^{1+d}$, $\alpha = 0, 1, 2, 3$.

The corresponding *gauge fields* are given by the expressions:

$$F_A = -\frac{i}{2} g F_{\alpha\beta} dx^\alpha \wedge dx^\beta, \ \text{with } F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + ig\left[A_\alpha, A_\beta\right] \tag{1.4.10}$$

$$F_B = -\frac{i}{2} g_* G_{\alpha\beta} t_0 dx^\alpha \wedge dx^\beta, \ \text{with } G_{\alpha\beta} = \partial_\alpha B_\beta - \partial_\beta B_\alpha. \tag{1.4.11}$$

Concerning the *Higgs matter field* ϕ, it is simply expressed by a $\mathbb{C}^2$-valued smooth function

$$\phi : \mathbb{R}^{1+3} \longrightarrow \mathbb{C}^2, \ \phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}; \tag{1.4.12}$$

and it is weakly coupled to the potential field by means of the covariant derivative

$$D_\alpha \phi = \partial_\alpha \phi - igA_\alpha^a t_a \phi - ig_* B_\alpha t_0 \phi, \ \alpha = 0, 1, 2, 3. \tag{1.4.13}$$

Over $\mathbb{C}^2$, we consider the standard inner product:

$$(\Phi, \Psi) = \Phi^\dagger \Psi = \bar{\varphi}_1 \psi_1 + \bar{\varphi}_2 \psi_2, \ \text{ for } \Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2. \tag{1.4.14}$$

While, with the identification (1.4.7), we may use the usual inner product (1.3.83) for the elements of the gauge algebra.

In this way, the *bosonic* sector of the electroweak theory is formulated according to the Lagrangean density

$$\begin{aligned} \mathcal{L}(\mathcal{A}, B, \varphi) = &-\frac{1}{4} g^2 \operatorname{Tr}\left(F_{\alpha\beta}\left(F^{\alpha\beta}\right)^\dagger\right) - \frac{1}{4} g_*^2 G_{\alpha\beta} G^{\alpha\beta} + \left(D^\alpha \phi\right)^\dagger \left(D_\alpha \phi\right) \\ &- \lambda\left(\varphi_0^2 - \phi^\dagger \phi\right)^2, \end{aligned} \tag{1.4.15}$$

with λ and φ_0 two positive parameters.

For $\omega^a, \xi : \mathbb{R}^{1+3} \to \mathbb{R}$, smooth functions ($a = 1, 2, 3$), and $f = e^{ig\omega^a t_a} \in SU(2)$, we check that the Lagrangean density in (1.4.15) is invariant under the gauge transformations:

$$\begin{aligned} \varphi &\longrightarrow e^{ig\omega^a t_a + ig_* \xi t_0} \varphi, \\ A_\alpha &\longrightarrow f A_\alpha f^{-1} + f \partial_\alpha f^{-1}, \\ B_\alpha &\longrightarrow B_\alpha + \partial_\alpha \xi. \end{aligned} \tag{1.4.16}$$

By virtue of such invariance, we can rewrite the theory in the unitary gauge defined by the condition

$$\phi = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \tag{1.4.17}$$

with φ a *real scalar* function.

In fact, in the unitary gauge, it is convenient to formulate the theory in terms of the new fields' configurations — W_α, P_α and Z_α — obtained as the linear combinations of the original fields as follows:

$$P_\alpha = B_\alpha \cos\theta + A_\alpha^3 \sin\theta, \tag{1.4.18}$$

$$Z_\alpha = -B_\alpha \sin\theta + A_\alpha^3 \cos\theta, \tag{1.4.19}$$

and

$$W_\alpha = \frac{1}{\sqrt{2}}\left(A_\alpha^1 + iA_\alpha^2\right), \tag{1.4.20}$$

$\alpha = 0, 1, 2, 3$ and $\theta \in (0, \frac{\pi}{2})$. To comprehend the role of the angle θ, we observe that W_α and Z_α are massive fields mediating short-range (weak) interactions, while P_α mediates long-range (electromagnetic) interactions (see [La]).

In terms of P_α and Z_α, the covariant derivative D_α takes the expression:

$$\begin{aligned} D_\alpha = \partial_\alpha - ig\left(A_\alpha^1 t_1 + A_\alpha^2 t_2\right) - iP_\alpha\left((g\sin\theta)t_3 + (g_*\cos\theta)\,t_0\right) \\ -iZ_\alpha\left((g\cos\theta)t_3 - (g_*\sin\theta)\,t_0\right). \end{aligned} \tag{1.4.21}$$

For P_α to mediate electromagentic interactions, we need the relative coefficient in (1.4.21) to correspond to the charge operator $eQ = e(t_3 + t_0)$, where e is the electron's charge. Consequently we derive: $e = g_* \cos\theta = g\sin\theta$. That is

$$e = \frac{gg^*}{\left(g^2 + g_*^2\right)^{\frac{1}{2}}} \text{ and } \cos\theta = \frac{g}{\left(g^2 + g_*^2\right)^{\frac{1}{2}}}, \tag{1.4.22}$$

$\theta \in (0, \frac{\pi}{2})$.

The angle θ is known as the Weinberg (mixing) angle, and from now on it will be fixed according to (1.4.22).

In this way, we see that

$$D_\alpha = \partial_\alpha - ig\left(A_\alpha^1 t_1 + A_\alpha^2 t_2\right) - iP_\alpha eQ - iZ_\alpha eQ', \tag{1.4.23}$$

with $Q' = (\cot\theta)t_3 - (\tan\theta)t_0$.

Consequently, in the unitary gauge where ϕ is assigned according to (1.4.27), we have

$$D_\alpha\phi = \begin{pmatrix} -\frac{i}{2}g\left(A_\alpha^1 - iA_\alpha^2\right)\varphi \\ \partial_\alpha\varphi + \frac{ig}{2\cos\theta}Z_\alpha\varphi \end{pmatrix}, \tag{1.4.24}$$

with φ a real scalar function.

Therefore, if we let $\mathcal{D}_\alpha = \partial_\alpha - igA^3_\alpha$ and set $P_{\alpha\beta} = \partial_\alpha P_\beta - \partial_\beta P_\alpha$, $Z_{\alpha\beta} = \partial_\alpha Z_\beta - \partial_\beta Z_\alpha$, then by direct inspection we can confirm that the Lagrangean density in (1.4.15), when expressed in the unitary gauge variables, takes the form:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\left(\mathcal{D}_\alpha W_\beta - \mathcal{D}_\beta W_\alpha\right)\overline{\left(\mathcal{D}^\alpha W^\beta - \mathcal{D}^\beta W^\alpha\right)} - \frac{1}{4}Z_{\alpha\beta}Z^{\alpha\beta} - \frac{1}{4}P_{\alpha\beta}P^{\alpha\beta}\\ &-\frac{1}{2}g^2\left(\left(\overline{W}^\alpha W_\alpha\right)^2 - W^\alpha W_\alpha\overline{W^\beta W_\beta}\right) - ig\left(Z^{\alpha\beta}\cos\theta + P^{\alpha\beta}\sin\theta\right)\overline{W}_\alpha W_\beta\\ &+\frac{1}{2}g^2\varphi^2\overline{W}^\alpha W_\alpha + \partial^\alpha\varphi\partial_\alpha\varphi + \frac{1}{4\cos^2\theta}g^2\varphi^2 Z^\alpha Z_\alpha - \lambda\left(\varphi_0^2 - \varphi^2\right)^2. \end{aligned} \quad (1.4.25)$$

To obtain selfdual electroweak vortex-type configurations, Ambjorn and Olesen (see [AO1], [AO2], and [AO3]) suggested that one take all magnetic excitation as confined in the third direction, according to the following vortex ansatz:

$$\begin{aligned} &A^a_0 = A^a_3 = B_0 = B_3 = 0,\\ &A^a_j = A^a_j(x^1, x^2),\ B_j = B_j(x^1, x^2),\ j = 1, 2, \end{aligned} \quad (1.4.26)$$

$a = 1, 2, 3$, and

$$\varphi = \varphi(x^1, x^2). \quad (1.4.27)$$

In addition, they assume that for a complex field W, there holds

$$W_1 = W,\ iW_2 = W; \quad (1.4.28)$$

that is,

$$-A^2_2 = A^1_1 = \sqrt{2}\mathcal{R}e(W),\ A^2_1 = A^1_2 = \sqrt{2}\mathcal{I}m(W). \quad (1.4.29)$$

We then find the following expression for the corresponding energy density

$$\begin{aligned}\mathcal{E} = &|\mathcal{D}_1 W + \mathcal{D}_2 W|^2 + \frac{1}{2}P^2_{12} + \frac{1}{2}Z^2_{12} - 2g(Z_{12}\cos\theta + P_{12}\sin\theta)|W|^2\\ &+ 2g^2|W|^4 + (\partial_i\varphi)^2 + \frac{1}{4\cos^2\theta}g^2\varphi^2 Z^2_j + g^2\varphi^2|W|^2 + \lambda(\varphi_0^2 - \varphi^2)^2.\end{aligned}$$

Notice that, under the ansatz (1.4.26)–(1.4.28), the invariance of $\mathcal{L}$ according to the transformations (1.4.16) is simply expressed by a residual $U(1)$-invariance, of the energy $\mathcal{E}$ defined above, under the gauge transformation:

$$\begin{aligned} &W \longrightarrow e^{i\xi}W, \qquad &&P_j \longrightarrow P_j + \tfrac{1}{e}\partial_j\xi\\ &Z_j \longrightarrow Z_j, &&\varphi \longrightarrow \varphi\ \ j = 1, 2, \end{aligned} \quad (1.4.30)$$

for any smooth function $\xi = \xi(x^1, x^2)$.

With the aim to attain selfduality, Ambjorn and Olesen in [AO1] observed that the above energy density $\mathcal{E}$ may be written in the form,

$$\begin{aligned}\mathcal{E} &= |\mathcal{D}_1 W + \mathcal{D}_2 W|^2 + \frac{1}{2}\left(P_{12} - \frac{g}{2\sin\theta}\varphi_0^2 - 2g\sin\theta|W|^2\right)^2 \\ &+ \frac{1}{2}\left(Z_{12} - \frac{g}{2\cos\theta}\left(\varphi^2 - \varphi_0^2\right) - 2g\cos\theta|W|^2\right)^2 + \left(\frac{g}{2\cos\theta}\varphi Z_j + \varepsilon_{jk}\partial_k\varphi\right)^2 \\ &+ \left(\lambda - \frac{g^2}{8\cos^2\theta}\right)\left(\varphi^2 - \varphi_0^2\right)^2 - \frac{g^2}{8\sin^2\theta}\varphi_0^4 + \frac{g\varphi_0^2}{2\sin\theta}P_{12} - \frac{g\varphi_0^2}{2\sin\theta}Z_{12} \\ &- \frac{g}{2\cos\theta}\partial_k\left(\varepsilon_{jk}Z_j\varphi^2\right). \end{aligned} \tag{1.4.31}$$

Therefore, for $\lambda \geq \frac{g^2}{8\cos^2\theta}$ we have

$$\mathcal{E} \geq -\frac{g^2}{8\sin^2\theta}\varphi_0^4 + \frac{g\varphi_0^2}{2\sin\theta}P_{12} - \frac{g\varphi_0^2}{2\sin\theta}Z_{12} - \frac{g}{2\cos\theta}\partial_k\left(\varepsilon_{jk}Z_j\varphi^2\right). \tag{1.4.32}$$

In addition, when the given parameters satisfy the "critical" condition

$$\lambda = \frac{g^2}{8\cos^2\theta}, \tag{1.4.33}$$

then the lower bound in (1.4.32) is saturated by the solutions of the following system of *selfdual equations*:

$$\begin{cases} \mathcal{D}_1 W + i\mathcal{D}_2 W = 0, \\ P_{12} = \frac{g}{2\sin\theta}\varphi_0^2 + 2g\sin\theta|W|^2, \\ Z_{12} = \frac{g}{2\cos\theta}\left(\varphi^2 - \varphi_0^2\right) + 2g\cos\theta|W|^2, \\ Z_j = -\frac{2\cos\theta}{g}\varepsilon_{jk}\partial_k\log\varphi, \;\; j = 1, 2. \end{cases} \tag{1.4.34}$$

As usual, (1.4.34) represents a first-order factorization of the Euler–Lagrange equations corresponding to $\mathcal{L}$ in (1.4.25). In particular, each solution of (1.4.34) gives rise to a critical point for $\mathcal{L}$. Note that in the unitary gauge variables, the real field φ never vanishes. Also observe that we can combine the last two equations in (1.4.34) and obtain,

$$-\Delta\log\varphi = \left(\frac{g}{2\cos\theta}\right)^2(\varphi_0^2 - \varphi^2) - g^2|W|^2. \tag{1.4.35}$$

Expression (1.4.32) implies that *planar* solutions of (1.4.34) may carry *infinite* energy. Therefore, in the planar case, the selfduality criterion yields to solutions which may not bare a physical meaning. Nonetheless, we know that the solutions of (1.4.34) appear in abundance and may be selected according to their decay property at infinity (see [SY3] and [ChT1]).

As we shall see, the appropriate boundary conditions to consider in this context are the periodic ones, as introduced by 'tHooft in a gauge-invariant situation (see ['tH2]). The corresponding periodic electroweak vortex-like configurations are known as *W-condensates*, and are physically interesting by virtue of their connection to the so-called Abrikosov's "mixed states" in superconductivity.

Since their presence was first predicted by Ambjorn and Olesen ([AO1], [AO2], and [AO3]) on the grounds of some numerical evidence, the existence of W-condensates has been established rigorously in [SY2] and [BT2]. We will discuss them in Chapter 7.

The analysis of selfdual electroweak vortex configurations actually can be extended to selfgravitating electroweak strings; they occur in the theory described above, when we also take into account the effect of gravity. In this situation, the metric tensor g is no longer fixed but part of the unknowns to be determined according to the coupled electroweak Einstein equations.

Again, it is possible to show that a selfdual regime is attained by the electroweak Einstein theory, when we allow the gravitational metric to vary in the class

$$ds^2 = \left(dx^0\right)^2 - \left(dx^3\right)^2 - e^{\eta}\left(\left(dx^1\right)^2 + \left(dx^2\right)^2\right), \tag{1.4.36}$$

with η the unknown conformal factor.

Thus, as before, for λ fixed according to the critical condition (1.4.33), and under the *string ansatz* according to which η is a function depending only on the variables (x^1, x^2), and when (1.4.26)–(1.4.29) hold, we find that selfgravitating strings (parallel along the x^3-axis) may be obatined by solving a set of selfdual equations which modify (1.4.34) as follows:

$$\begin{cases} \mathcal{D}_1 W + i\mathcal{D}_2 W = 0, \\ P_{12} = \frac{g}{2\sin\theta}\varphi_0^2 e^{\eta} + 2g\sin\theta|W|^2, \\ Z_{12} = \frac{g}{2\cos\theta}\left(\varphi^2 - \varphi_0^2\right)e^{\eta} + 2g\cos\theta|W|^2, \\ Z_j = -\frac{2\cos\theta}{g}\varepsilon_{jk}\partial_k \log\varphi, \ \ j = 1, 2, \end{cases} \tag{1.4.37}$$

with η satisfying Einstein's equations that, in this setting, reduce to

$$-\Delta\left(\frac{\eta}{8\pi G}\right) = \frac{g\varphi_0^2}{\sin\theta}P_{12} + \frac{g}{\cos\theta}\left(\varphi^2 - \varphi_0^2\right)Z_{12} + 4|\nabla\varphi|^2, \tag{1.4.38}$$

where G is the gravitational constant.

We refer the reader to [Y1] for the details. Here we limit ourselves to observing that the energy density, corresponding to solution of (1.4.37), takes the form:

$$\mathcal{E} = -\frac{g^2\varphi_0^4}{8\sin^2\theta} + \frac{g\varphi_0^2}{2\sin\theta}e^{-\eta}P_{12} - \frac{g\varphi_0^2}{2\cos\theta}e^{-\eta}Z_{12} - \frac{g}{2\cos\theta}e^{-\eta}\partial_k(\varepsilon_{jk}Z_j\varphi^2). \tag{1.4.39}$$

As a matter of fact, we can use (1.4.37) to express $\mathcal{E}$ equivalently as follows:

$$\begin{aligned} \mathcal{E} &= -\frac{g^2\varphi_0^4}{8\sin^2\theta} + \frac{g\varphi_0^2}{2\sin\theta}e^{-\eta}P_{12} + \frac{g}{2\cos\theta}e^{-\eta}Z_{12}\left(\varphi^2 - \varphi_0^2\right) + 2e^{-\eta}|\nabla\varphi|^2 \\ &= \frac{g^2\varphi_0^4}{8\sin^2\theta} + \frac{g^2}{4\cos^2\theta}\left(\varphi^2 - \varphi_0^2\right)^2 + g^2\varphi^2|W|^2 e^{-\eta} + 2e^{-\eta}|\nabla\varphi|^2. \end{aligned} \tag{1.4.40}$$

Therefore, contrary to the previous case, it makes good sense now to consider *planar* selfgravitating electroweak strings (parallel in the x^3-direction) satisfying the finite energy condition

$$\int_{\mathbb{R}^2} \mathcal{E} e^{\eta} < +\infty, \tag{1.4.41}$$

which can be verified by requiring adeguate behavior of η at infinity.

It is interesting to note that (1.4.39) leads also to a geometrical property about the Gauss curvature $K_\eta = -\frac{1}{2} e^{-\eta} \Delta \eta$ of the Riemann surface $(\mathbb{R}^2, e^{\eta}\delta_{jk})$. Indeed, by means of Einstein's equation, K_η relates to $\mathcal{E}$ by the identity

$$K_\eta = 8\pi G \mathcal{E} + \Lambda,$$

with Λ the cosmological constant, which (for consistency) must be specified as

$$\Lambda = \frac{\pi G g^2 \varphi_0^4}{\sin^2 \theta}.$$

So, we can interpret the finite energy condition (1.4.41) as (almost) equivalent to a finite total Gauss curvature property.

Rigorous analytical results concerning the existence of selfgravitating electroweak strings, including solutions to (1.4.37), (1.4.38), and (1.4.41) (with finite total Gauss curvature) have been established recently in [ChT2], and will be discussed in Chapter 7.

2

Elliptic Problems in the Study of Selfdual Vortex Configurations

2.1 Elliptic formulation of the selfdual vortex problems

The examples of selfdual problems discussed in the previous chapter all share a common equation (see (2.1.1) below), which can be viewed as a gauge-invariant version of the Cauchy–Riemann equation.

Following an approach introduced by Taubes (cf. [JT]), we see next how to use such a property in order to eliminate the gauge invariance from the selfdual equations and formulate them in terms of elliptic problems of the Liouville-type, whose analysis will be the main objective of our study.

To be more precise, let $\phi \in \mathbb{C}$ be a smooth complex-valued function defined in $\mathbb{R}^2$ (to be identified with $\mathbb{C}$ when necessary), and $\mathcal{A} = (A_j)_{j=1,2;}$ be a smooth real vector field such that,

$$D_{\pm}\phi := (\partial_1 \pm i\partial_2)\phi - i(A_1 \pm iA_2)\phi = 0 \text{ in } \mathbb{R}^2. \tag{2.1.1}$$

Since (2.1.1) is invariant under the abelian-gauge transformation,

$$\phi \longrightarrow e^{i\omega}\phi,\ A_j \longrightarrow A_j + \partial_j\omega,\ j = 1, 2 \tag{2.1.2}$$

(for any given smooth real function ω defined over $\mathbb{R}^2$), we may suppose $\mathcal{A}$ to be specified according to the Coulumb gauge, where $\mathcal{A}$ defines a divergence-free field, namely, $\partial_1 A_1 + \partial_2 A_2 = 0$. Thus, if we let η be a real function such that,

$$\nabla\eta = \pm(-A_2, A_1), \tag{2.1.3}$$

then $\psi = e^{-\eta}\phi$ satisfies $(\partial_1 \pm i\partial_2)\psi = 0$. So according to whether we choose the $+$ or $-$ sign, we find that ψ or $\overline{\psi}$ is holomorphic.

Therefore, if ϕ is a non-trivial solution of (2.1.1), then ϕ admits only an isolated number of zeroes with integral multiplicity. From the point of view of the vortex problem, such zeroes play the role of defects and are responsible for the occurence of non-trivial phenomena.

Assuming that $\{z_1, \ldots, z_N\}$ are the zeroes of ϕ, repeated according to their multiplicity, and setting

$$h(z) = e^{-\eta(z)}\phi(z)\left(\prod_{j=1}^{N}\left(z - z_j\right)\right)^{-1}$$

or

$$h(z) = e^{-\eta(z)}\overline{\phi}(z)\left(\prod_{j=1}^{N}\overline{\left(z - z_j\right)}\right)^{-1},$$

we see that h defines a never vanishing holomorphic function. Furthermore, by the change of gauge

$$\phi \longrightarrow |h|h^{-1}\phi,$$

we find that ϕ takes the form

$$\phi(z) = |\phi(z)|e^{\pm i\sum_{j=1}^{N}\mathrm{Arg}(z-z_j)} \tag{2.1.4}$$

with

$$|\phi(z)| = e^{\eta}|h(z)|\prod_{j=1}^{N}\left|z - z_j\right|. \tag{2.1.5}$$

In this gauge, equation (2.1.1) reads as

$$\begin{aligned} \partial_2 \log|\phi| + \partial_1\left(\sum_{j=1}^{N}\mathrm{Arg}\left(z - z_j\right)\right) &= \pm A_1 \\ \partial_1 \log|\phi| - \partial_2\left(\sum_{j=1}^{N}\mathrm{Arg}\left(z - z_j\right)\right) &= \mp A_2, \end{aligned} \tag{2.1.6}$$

and gives a well-defined smooth expression for the components A_1 and A_2 in terms of the (gauge-invariant quantity) $|\phi|$. Furthermore, from (2.1.5) and (2.1.3) and the fact that $\log|h|^2$ is a harmonic function, we deduce that,

$$\begin{aligned} -\Delta\log|\phi|^2 &= -2\Delta\eta - 2\Delta\log|h|^2 - 4\pi\sum_{j=1}^{N}\delta_{z_j} \\ &= \pm 2F_{12} - 4\pi\sum_{j=1}^{N}\delta_{z_j}. \end{aligned}$$

Here δ_p denotes the Dirac measure with a pole at $p \in \mathbb{R}^2$.

At this point, we can use the remaining selfdual equations to determine $\log|\phi|^2$ (and hence $|\phi|$) as a solution of an elliptic problem.

In this way, we have eliminated completely the fastidious gauge invariance from the selfdual equations and reduced their study to the solvability of some elliptic problems involving only gauge-invariant quantities.

To be more precise, let us start to discuss the abelian case, where ϕ defines the Higgs (matter) field, and set

$$u = \log |\phi|^2. \tag{2.1.7}$$

For a given set of points $\{z_1, \ldots, z_N\}$, repeated according to their multiplicity, we can take advantage of (2.1.4), (2.1.6), and (2.1.8), to define

$$\phi(z) = e^{\frac{1}{2}u(z) \pm i \sum_{j=1}^N \operatorname{Arg}(z-z_j)}, \tag{2.1.8}$$

and

$$\begin{aligned} A_1 &= \pm\left(\frac{1}{2}\partial_2 u + \partial_1\left(\sum_{j=1}^N \operatorname{Arg}(z-z_j)\right)\right), \\ A_2 &= \mp\left(\frac{1}{2}\partial_1 u - \partial_2\left(\sum_{j=1}^N \operatorname{Arg}(z-z_j)\right)\right). \end{aligned} \tag{2.1.9}$$

Based on the arguments above, $(\phi, \mathcal{A})$ with $\mathcal{A} = -iA_\alpha dx^\alpha$ defines a vortex solution for the *abelian Maxwell–Higgs selfdual* equations (1.2.25), (1.2.26), and (1.2.27), provided we take ϕ in (2.1.8) and A_1 and A_2 in (2.1.9) such that u satisfies

$$-\Delta u = 1 - e^u - 4\pi \sum_{j=1}^N \delta_{z_j}, \tag{2.1.10}$$

and let,

$$A_0 = 0. \tag{2.1.11}$$

Similarly, we obtain a vortex solution for the pure *Chern–Simons–Higgs selfdual* equations (1.2.45), provided we now take u in (2.1.8) and (2.1.9) to satisfy

$$-\Delta u = \frac{4}{k^2}e^u(\nu^2 - e^u) - 4\pi \sum_{j=1}^N \delta_{z_j}, \tag{2.1.12}$$

and recalling (1.2.46), let

$$A_0 = \pm\frac{1}{k}(\nu^2 - e^u). \tag{2.1.13}$$

Similarly, to derive a vortex solution for the *Maxwell–Chern–Simons–Higgs selfdual* equations (1.2.63), we need to take u and the neutral scalar field N to satisfy

$$\begin{cases} -\Delta u = 2q^2\left(kN - e^u\right) - 4\pi \displaystyle\sum_{j=1}^N \delta_{z_j}, \\ -\dfrac{1}{q^2}\Delta N = 2e^u\left(\dfrac{\nu^2}{k} - N\right) + kq^2(e^u - kN), \end{cases} \tag{2.1.14}$$

and recalling (1.2.61), let

$$A_0 = \pm\left(\frac{\nu^2}{k} - N\right). \tag{2.1.15}$$

Note that for a vortex solution constructed in this manner, the Higgs field vanishes exactly at $\{z_1, \cdots, z_N\}$ according to the given multiplicity. Furthermore, for the solution u of any of the elliptic equations introduced above holds the decomposition

$$u(z) = \sum_{j=1}^{N} \log \left|z - z_j\right|^2 + \text{ smooth function}, \tag{2.1.16}$$

and so we may check that the potential field components A_1 and A_2 in the right-hand side of (2.1.9) extend smoothly at $\{z_1, \ldots, z_N\}$, the zeroes set of ϕ.

Analogously, for the non-abelian pure Chern–Simons model in (1.3.99), (1.3.100), and (1.3.101), we may derive a selfdual vortex $(\mathcal{A}, \phi)$ under the ansatz

$$\mathcal{A} = A_\alpha dx^\alpha \text{ with } A_\alpha = -iA_\alpha^a H_a,\ A_\alpha^a \in \mathbb{R} \text{ and } \phi = \phi^a E_a \quad \phi^a \in \mathbb{C}, \tag{2.1.17}$$

where H_a and E_a, for $a = 1, \ldots, r$ are respectively the Cartan algebra generators and the simple root operators for the (semisimple) gauge group G with rank r. For the given integers $N_a \in \mathbb{N}$ and the set of points $\{z_1^a, \ldots, z_{N_a}^a\}$ (repeated according to their multiplicity) set

$$\phi^a = e^{\frac{1}{2}u_a \pm i \sum_{j=1}^{N_a} \operatorname{Arg}\left(z - z_j^a\right)}, \tag{2.1.18}$$

$a = 1, \ldots, r$. Let the components A_j^a, $j = 1, 2$ be defined in terms of the invertible Cartan matrix, $K = (K_{ab})$ in (1.3.79), through the identity:

$$\begin{aligned} K_{ba}A_1^b &= \pm\left(\frac{1}{2}\partial_2 u_a + \partial_1 \sum_{j=1}^{N_a} \operatorname{Arg}\left(z - z_j^a\right)\right), \\ K_{ba}A_2^b &= \mp\left(\frac{1}{2}\partial_1 u_a - \partial_2 \sum_{j=1}^{N_a} \operatorname{Arg}\left(z - z_j^a\right)\right), \\ a &= 1, \ldots, r. \end{aligned} \tag{2.1.19}$$

Recalling (1.3.124), we can also obtain the component A_0^a by means of the relation:

$$K_{ba}A_0^b = \pm\frac{1}{k}\left(\nu^2 - K_{ba}e^{u_b}\right),\ a = 1, \ldots, r. \tag{2.1.20}$$

In this way, we determine a vortex solution for the non-abelian Chern–Simons selfdual equations (1.3.115) (reduced to (1.3.123) via the ansatz (2.1.17)) provided $(u_a)_{a=1,\ldots,r}$ defines a solution for the elliptic system:

$$-\Delta u_a = \frac{4}{k^2}\left(\nu^2 K_{ba} e^{u_b} - K_{ba} e^{u_b} K_{cb} e^{u_c}\right) - 4\pi \sum_{j=1}^{N} \delta_{z_j^a}. \tag{2.1.21}$$

If $G = SU(n+1)$, then the rank $r = n$ and the Cartan matrix $K = (K_{ab})$ is explicitly given by (1.3.79). Hence, in this case, we can interpret the system (2.1.21) as the natural extension of the single Chern–Simons equation (2.1.12) to a Toda lattice system coupled by the $SU(n+1)$ Cartan matrix (1.3.79).

Concerning *electroweak vortices*, obtained under the ansatz (1.4.26)–(1.4.29) as solutions of the selfdual equations (1.4.34), we can use a similar approach, and for assigned points $\{z_1, \ldots, z_N\}$ (repeated according to their multiplicity) we set,

$$W = e^{\frac{1}{2}u + \sum_{j=1}^{N} \mathrm{Arg}\left(z - z_j^a\right)}. \tag{2.1.22}$$

We verify the first equation in (1.4.34), namely,

$$\mathcal{D}_1 W + i\mathcal{D}_2 W = (\partial_1 + i\partial_2)\, W - ig\left(A_1^3 + iA_2^3\right) = 0$$

provided u satisfies

$$-\Delta u = 2g\left(\partial_1 A_2^3 - \partial_2 A_1^3\right) = 2g\,(P_{12}\sin\theta + Z_{12}\cos\theta) - 4\pi \sum_{j=1}^{N} \delta_{z_j}$$

where recalling (1.4.18) and (1.4.19), we have taken

$$\begin{aligned} A_1^3 &= \frac{1}{g}\left(\frac{1}{2}\partial_2 u + \partial_1 \sum_{j=1}^{N} \mathrm{Arg}\left(z - z_j\right)\right), \\ A_2^3 &= -\frac{1}{g}\left(\frac{1}{2}\partial_1 u - \partial_2 \sum_{j=1}^{N} \mathrm{Arg}\left(z - z_j\right)\right), \end{aligned} \tag{2.1.23}$$

defined smoothly through the points z_j, $j = 1, \ldots, N$.

Furthermore, taking into account the last equation in (1.4.34) and (1.4.35), we let,

$$\varphi^2 = e^{v} \text{ and } Z_j = -\frac{\cos\theta}{g}\varepsilon_{jk}\partial_k v. \tag{2.1.24}$$

Thus, using (2.1.22), (2.1.23), and (2.1.24), we obtain an electroweak vortex configuration solution of the selfdual equation (1.4.34), with W vanishing exactly at $\{z_1, \ldots, z_N\}$, provided the pair (u, v) satisfies:

$$\begin{cases} -\Delta u = 4g^2 e^u + g^2 e^v - 4\pi \displaystyle\sum_{j=1}^{N} \delta_{z_j}, \\ -\Delta v = -2g^2 e^u - \dfrac{g^2}{2\cos^2\theta} e^v + \dfrac{g^2\varphi_0^2}{2\cos^2\theta}. \end{cases} \tag{2.1.25}$$

Observe that the fields B_j and P_j may be recovered from A_j^3 in (2.1.23) and from Z_j in (2.1.24) via (1.4.18) and (1.4.19), for $j = 1, 2$.

Analogously, we proceed to derive *selfgravitating electroweak strings*, which include the conformal factor η in the selfdual equations given in (1.4.37) and (1.4.38). As above, we see that this case reduces to solving the following elliptic system:

$$\begin{cases} -\Delta u = 4g^2 e^u + g^2 e^{v+\eta} - 4\pi \sum_{j=1}^{N} \delta_{z_j}, \\ -\Delta v = \dfrac{g^2}{2\cos^2\theta}\left(\varphi_0^2 - e^v\right) e^\eta - 2g^2 e^u, \\ -\Delta\left(\dfrac{\eta}{8\pi G}\right) = \dfrac{g^2}{2}\left(\dfrac{\left(e^v - \varphi_0^2\right)^2}{\cos^2\theta} + \dfrac{\varphi_0^4}{\sin^2\theta}\right) e^\eta + 2g^2 e^{u+v} + |\nabla v|^2 e^v. \end{cases} \tag{2.1.26}$$

Thus, for the selfdual models considered above, we can obtain vortex configurations, with a prescribed set of zeroes for the complex field, by solving certain elliptic problems.

At this point, it is important to be more specific about the boundary conditions. Clearly our choice of boundary conditions must be significative from the point of view of the physical applications, as for example in identifying relevant quantities such as the total energy, the magnetic flux, and the electric charge.

In this respect, the first set of boundary conditions we shall consider, concerns with *planar* selfdual vortices, where the selfdual equations (or equivalently, the corresponding elliptic problems) are examined in the whole plane $\mathbb{R}^2$ under suitable decay assumptions at infinity, that guarantee finite total energy.

The second set of boundary conditions we consider pertains to *periodicity*. Note that periodic patterns of vortex configurations have been observed experimentally to form in superconductivity, and that they are known as the Abrikosov's "mixed states," in view of Abrikosov's pioneering work about superconductors (cf. [Ab]) where such configurations were first predicted.

Therefore, for the (gauge-invariant) fields we require the 't Hooft periodic boundary conditions (cf. ['tH2]) over the cell domain

$$\Omega = \left\{ z = s_1 \mathbf{a}^1 + s_2 \mathbf{a}^2 \in \mathbb{R}^2,\ 0 < s_j < 1,\ j = 1, 2 \right\} \tag{2.1.27}$$

with $\mathbf{a}^1$ and $\mathbf{a}^2$ two linearly independent vectors in $\mathbb{R}^2$, and with the boundary component

$$\Gamma_k = \left\{ z = s_k \mathbf{a}^k,\ 0 < s_k < 1 \right\} \subset \partial\Omega,\ k = 1, 2.$$

Concerning *abelian periodic vortices*, we require the Higgs field $\phi = \phi(z)$, the component $A_\alpha,\ \alpha = 0, 1, 2$ of the potential field $\mathcal{A}$ and, when present, the neutral scalar field N to satisfy

$$\begin{aligned} \phi\left(z + \mathbf{a}^k\right) &= e^{i\omega_k(z)}\phi(z), \\ A_j\left(z + \mathbf{a}^k\right) &= A_j(z) + \partial_j \omega_k(z),\ j = 1, 2, \\ A_0\left(z + \mathbf{a}^k\right) &= A_0(z), \\ N\left(z + \mathbf{a}^k\right) &= N(z), \end{aligned} \tag{2.1.28}$$

for $z \in \Gamma_1 \cup \Gamma_2 \setminus \Gamma_k$, where ω_k is an arbitrary function smoothly defined in a neighborhood of $\Gamma_1 \cup \Gamma_2 \setminus \Gamma_k$, and $k = 1, 2$. For simplicity, we set $\omega_k(s_1, s_2) = \omega_k(s_1 \mathbf{a}^1 + s_2 \mathbf{a}^2)$.

A first interesting consequence of the boundary conditions (2.1.28) is a "quantization" property of the magnetic flux through Ω:

$$\Phi = \int_\Omega F_{12}.$$

Indeed, since ϕ is single valued in Ω, its face shift around $\partial\Omega$ must coincide with an integral multiple of 2π. In fact, by taking into account the first condition in (2.1.28) and the expressions (2.1.8) and (2.1.16) in Ω, we find that ω_k must satisfy

$$\omega_1(0, 1^-) - \omega_1(0, 0^+) - (\omega_2(1^-, 0) - \omega_2(0^+, 0)) = \pm 2\pi N, \tag{2.1.29}$$

with N the number of (prescribed) zeroes (counted with multiplicity) of ϕ in Ω.

Since,

$$\Phi = \int_\Omega F_{12} = \int_{\partial\Omega} \left(A_1\, dx^1 + A_2\, dx^2 \right) = -\int_0^1 \partial_1 \omega_2(s_1, 0)\, ds_1 + \int_0^1 \partial_2 \omega_1(0, s_2)\, ds_2,$$

from the above identity (2.1.29), we find

$$\Phi = \pm 2\pi N. \tag{2.1.30}$$

In other words, the total magnetic flux through Ω can only take a value corresponding to an integral multiple of 2π, determined according to the topological degree of ϕ in Ω.

This "quantization" property is shared by the static *total energy* $E = \int_\Omega \mathcal{E}$ and the *electric charge* $Q = \int_\Omega J^0$. In fact, by virtue of the boundary condition (2.1.28), the spatial divergence terms involved in $\mathcal{E}$ drop out, and for the *abelian Maxwell–Higgs model* we find

$$E = \pm \frac{1}{2} \int_\Omega F_{12} = \pi N, \qquad Q = \int_\Omega J^0 = 0, \tag{2.1.31}$$

as a consequence of (1.2.24) and (1.2.14) (applied at $\mu = 0$), respectively.

Analogously, for the *Chern–Simons model* we have

$$E = \pm \nu^2 \int_\Omega F_{12} = 2\pi \nu^2 N, \quad Q = \int_\Omega J^0 = k \int_\Omega F_{12} = \pm 2\pi k N \tag{2.1.32}$$

as a consequence of (1.2.42) and (1.2.36), respectively.

While, for the *Maxwell-Chern–Simons–Higgs model* we find

$$\begin{aligned} E &= \pm \frac{\nu^2}{k} \int_\Omega J^0 = \pm \frac{\nu^2}{k} \int_\Omega \left(-\frac{1}{q^2} \Delta A_0 + k F_{12} \right) = \nu^2 \int_\Omega F_{12} = 2\pi \nu^2 N, \\ Q &= \int_\Omega J^0 = \int_\Omega \left(-\frac{1}{q^2} \Delta A_0 + k F_{12} \right) = k \int_\Omega F_{12} = \pm 2\pi k N, \end{aligned} \tag{2.1.33}$$

as a consequence of (1.2.57) and (1.2.59), respectively.

Concerning the non-abelian Chern–Simons problem (1.3.123), we impose analogous periodic boundary conditions that hold componentwise, according to the gauge invariance (1.3.127) and (1.3.128) which is valid in this case. More precisely, we require

$$\begin{aligned} \phi^a\left(z+\mathbf{a}^k\right) &= e^{iK_{ba}\omega_k^b(z)}\phi(z), \\ A_j^a\left(z+\mathbf{a}^k\right) &= A_j^a(z)+\partial_j\omega_k^a(z),\ \ j=1,2 \\ A_0^a\left(z+\mathbf{a}^k\right) &= A_0^a(z), \end{aligned} \tag{2.1.34}$$

for $z \in \Gamma_1 \cup \Gamma_2 \setminus \Gamma_k$, and where ω_k^a is an arbitrary function smoothly defined in a neighborhood of $\Gamma_1 \cup \Gamma_2 \setminus \Gamma_k$, where $k = 1, 2$ and $a = 1, \ldots, r$.

Exactly as above, the given boundary conditions yield to a suitable "quantization" property involving the component of the magnetic flux and the electric charge through Ω defined respectively as follows:

$$\Phi_a = \int_\Omega F_{12}^a \text{ and } Q_a = \int_\Omega J_0^a, \quad a = 1, \ldots, r. \tag{2.1.35}$$

Indeed, setting

$$\hat{\omega}_k^a(s_1, s_2) = K_{ba}\omega_k^b(s_1\mathbf{a}^1 + s_2\mathbf{a}^2),$$

we can use (2.1.18) to evaluate the phase shift of ϕ^a around $\partial\Omega$ and find

$$\hat{\omega}_1^a(0,1^-) - \hat{\omega}_1^a(0,0^+) - (\hat{\omega}_2^a(1^-,0) - \hat{\omega}_2^a(0^+,0)) = \pm 2\pi N_a,$$

and this implies

$$\sum_{b=1}^r K_{ba}\Phi_b = \pm 2\pi N_a.$$

By virtue of (1.3.124) and (1.3.126), we obtain the following "quantization" rules:

$$\Phi_a = \pm 2\pi \sum_{b=1}^r \left(K^{-1}\right)_{ba} N_b; \quad Q_a = k\Phi_a = \pm 2\pi k \sum_{b=1}^r \left(K^{-1}\right)_{ba} N_b; \tag{2.1.36}$$

$$E = \int_\Omega \mathcal{E} = \pm \nu^2 \sum_{a=1}^r \Phi_a = 2\pi\nu^2 \sum_{a,b=1}^r \left(K^{-1}\right)_{ba} N_b. \tag{2.1.37}$$

Here again the "periodic" boundary conditions (2.1.34) allow us to ignore the total spatial divergence term in (1.3.126).

Finally, concerning *periodic electroweak vortices*, we take into account the corresponding gauge invariance (1.4.30), and require

$$\begin{cases} W\left(z+\mathbf{a}^k\right) = e^{i\omega_k(z)}W(z), \\ P_j\left(z+\mathbf{a}^k\right) = P_j(z) + \dfrac{1}{e}\partial_j\omega_k, \\ Z_j\left(z+\mathbf{a}^k\right) = Z_j(z), \\ \varphi\left(z+\mathbf{a}^k\right) = \varphi(z), \end{cases} \tag{2.1.38}$$

for $z \in \Gamma_1 \cup \Gamma_2 \setminus \Gamma_k$, and where ω_k is an arbitrary function smoothly defined in a neighborhood of $\Gamma_1 \cup \Gamma_2 \setminus \Gamma_k$ for $k = 1, 2$.

Exactly as above, we derive a "quantization" property for the flux of the field $(P_j)_{j=1,2}$ through Ω as,

$$\Phi = \int_\Omega P_{12} = \frac{2\pi}{e} N, \tag{2.1.39}$$

where N is the number of the (prescribed) zeroes (counted with multiplicity) of W in Ω. Furthermore, by means of (1.4.31), we can take advantage of (2.1.38) to obtain the following expression for the corresponding total energy:

$$\begin{aligned} E = \int_\Omega \mathcal{E} &= \frac{g\varphi_0^2}{2\sin\theta}\left(\int_\Omega P_{12} - \frac{g\varphi_0^2}{4\sin\theta}|\Omega|\right) \\ &= \frac{g\varphi_0^2}{2\sin\theta}\left(\frac{2\pi N}{e} - \frac{g\varphi_0^2}{4\sin\theta}|\Omega|\right). \end{aligned} \tag{2.1.40}$$

Similarly, we can handle periodic electroweak strings by supplementing the boundary conditions (2.1.38) with a periodicity assumption on the conformal factor:

$$\eta\left(z + \mathbf{a}^k\right) = \eta(z), \quad \forall z \in \Gamma_1 \cup \Gamma_2 \setminus \Gamma_k \text{ and } k = 1, 2. \tag{2.1.41}$$

Clearly, the "quantization" property (2.1.39) continues to hold in this case, and for the total energy we find

$$E = \int_\Omega \mathcal{E} e^\eta = \frac{g\varphi_0^2}{2\sin\theta}\left(\frac{2\pi N}{e} - \frac{g\varphi_0^2}{4\sin\theta}\int_\Omega e^\eta\right), \tag{2.1.42}$$

as a consequence of (1.4.39).

The 'tHooft periodic boundary conditions reduce to the usual periodicity for gauge-invariant quantities (e.g., $|\phi|$, F_{12}, etc.). In order to obtain periodic vortices, we will therefore need to attack the reduced elliptic problems under periodic boundary conditions, or equivalently, over the flat 2-torus.

Actually, from a mathematical point of view, one may wish to consider the given elliptic problems more generally over a compact surface. Nonetheless, the nature of these problems is such that the torus comes to play a special role, somewhat similar to that played by the 2-sphere in the prescribed Gauss curvature problem (cf. [ChY3]).

In conclusion, we have seen that, by prescribing the set of zeroes for the complex Higgs field (defined over $\mathbb{R}^2$ or over the periodic cell domain Ω), we can reduce the study of the selfdual vortex problem to the search of solutions for an elliptic equation (often in system form) over $\mathbb{R}^2$ (to yield planar vortices), or under periodic boundary conditions on $\partial\Omega$, (to yield periodic vortices).

Because of such a central role, we shall refer to a zero of the complex Higgs field as a *vortex point* and to its total number (counted with multiplicity) as the *vortex number*.

For the Ginzburg–Landau model in superconductivity, the vortex points identify the locus where the magnetic field is maximal and is expected to "localize."

As already mentioned, in this case the vortex number bares a topological information since it coincides with the Brouwer degree of the Higgs field and may be viewed as an element of the homotopy group $\pi_1(U(1)) = \mathbb{Z}$. Accordingly, the Maxwell–Higgs periodic vortices have been classified topologically in terms of such homotopy classes, and in this way completely characterized. See [Ta1], [Ta2], and [JT] for the planar case, and more generally, see [Bra1], [Bra2], [Ga1], [Ga2], [Ga3], [Hi], [HJS], and [WY].

Surprisingly, different characteristics have been observed for selfdual Chern–Simons vortices, where "non-topological" configurations are known to occur as we shall discuss in Chapter 3.

2.2 The solvability of Liouville equations

In the previous section, we have seen how the study of the vortex problems may be reduced to the solvability of suitable elliptic equations (or systems) involving exponential nonlinearities.

In order to analyze such elliptic problems, we start to review the Liouville equation. This provides an instructive example for the understanding of more general differential equations involving exponential nonlinearities.

To this purpose, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by means of the transformation $(x, y) \in \mathbb{R}^2 \to z = x + iy \in \mathbb{C}$. Moreover, in $\mathbb{C}$, we consider the usual scalar product:

$$\langle z, w\rangle = \mathcal{R}e(z, \overline{w}).$$

Accordingly, we express the Laplacian operator in terms of the complex differential operators $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ as follows:

$$\Delta = 4\partial_z\partial_{\overline{z}}.$$

Such complex notation is particularly useful in describing solutions to the (local) Liouville equation

$$-\Delta u = e^u \quad \text{in } D, \tag{2.2.1}$$

with D an open regular subset of $\mathbb{R}^2$ (i.e., $\mathbb{C}$).

For any holomorphic function $f = f(z)$ in D there holds:

$$\Delta \log\left(1 + |f(z)|^2\right) = 4\frac{\left|f'(z)\right|^2}{\left(1 + |f(z)|^2\right)^2} \tag{2.2.2}$$

(see [Ne]).

Thus, if f is univalent in D, then

$$u(z) = \log \frac{8|f'(z)|^2}{\left(1 + |f(z)|^2\right)^2} \tag{2.2.3}$$

satisfies (2.2.1).

Actually the expression (2.2.3) gives a full characterization of *all* solutions to (2.2.2), as was shown by Liouville in [Lio]. If we perturb (2.2.1) as

$$-\Delta u = e^u - 4\pi\alpha\delta_{z_0} \text{ in } D, \tag{2.2.4}$$

with $\alpha > 0$ and δ_{z_0} representing the Dirac measure with a pole at $z_0 \in D$, then it is possible to obtain an analogous characterization to the solutions of (2.2.4) by means of a Liouville-type formula valid for the punctured disk, as obtained by Chou–Wan in [CW].

For this case according to (2.2.2), f' or $(\frac{1}{f})'$ must vanish in D exactly at the point z_0 with multiplicity α. So for $\alpha \notin \mathbb{N}$, we must allow f to be multivalued in D. More precisely, on the basis of [CW], Bartolucci–Tarantello have shown in [BT1] that every solution of (2.2.4) takes the form of (2.2.3) with an univalent function f in $D \setminus \{z_0\}$ given by *one* of the following expressions:

(i) $$f(z) = (z - z_0)^{\alpha+1}\psi(z) + a, \text{ with } a \in \mathbb{C} \text{ and } a = 0 \text{ if } \alpha \notin \mathbb{N}; \tag{2.2.5}$$

(ii) $$f(z) = \frac{\psi(z)}{(z - z_0)^{\alpha+1}}; \tag{2.2.6}$$

(iii) limited to the case $\alpha = m - \frac{1}{2}$ with $m \in \mathbb{N}$: $f(z) = e^{(z-z_0)^{\alpha+1}\psi(z)}$, (2.2.7)

where ψ is a *holomorphic* function in D satisfying $\psi(z_0) \neq 0$.

See [BT1] for details.

Based on the preceding observations, the equation exhibit different classes of solutions. For instance, in the case of the planar singular Liouville equation (we take $z_0 = 0$ for simplicity):

$$-\Delta u = e^u - 4\pi\alpha\delta_{z=0} \text{ in } \mathbb{R}^2 \tag{2.2.8}$$

by choosing $f(z) = \mu z^{\alpha+1}$, $\mu \neq 0$, and setting $\lambda = \mu^2$ we obtain

$$u(z) = \log \frac{8\lambda(\alpha+1)^2|z|^{2\alpha}}{\left(1 + \lambda|z|^{2(\alpha+1)}\right)^2}, \ \forall\lambda > 0, \tag{2.2.9}$$

which solves (2.2.8).

Or, let $f(z) = \mu z^{\alpha+1}e^{g(z)}$ with $g(z) = (\alpha+1)\int_0^z \frac{e^\xi - 1}{\xi}\,d\xi$ and $\mu \neq 0$; then for $\lambda = \mu^2$, we also see that,

$$u(z) = \log \frac{8\lambda(\alpha+1)^2|z|^{2\alpha}|e^{g(z)+z}|^2}{\left(1 + \lambda|z|^{2(\alpha+1)}|e^{g(z)}|^2\right)^2}, \ \forall\lambda > 0 \tag{2.2.10}$$

solves (2.2.8).

The presence of the free parameter λ in (2.2.9) and (2.2.10) can be justified by the scale invariance of (2.2.8) under the transformation:

$$u(z) \longrightarrow u_\mu(z) := u(\mu z) + 2\log\mu, \ \forall\mu > 0.$$

Such an invariance will be at the origin of "concentration" phenomena for the solution sequences of Liouville-type equations, as will be discussed in great detail in Chapter 5.

As already shown by (2.2.5), when $\alpha = N \in \mathbb{Z}^+$, we gain an additional free (complex) parameter. In fact, we can generalize (2.2.9) in this case by taking $f(z) = \mu(z^{N+1} + \xi),\ \forall \xi \in \mathbb{C}$, and then set $\lambda = \mu^2$ to obtain the solution:

$$u(z) = \log \frac{8\lambda(N+1)^2|z|^{2N}}{\left(1 + \lambda|z^{N+1} + \xi|^2\right)^2},\ \forall \lambda > 0,\ \xi \in \mathbb{C}. \tag{2.2.11}$$

Notice that for $\alpha = N = 0$, the free parameter $\xi \in \mathbb{C}$ is due only to the translation invariance of (2.2.8).

If instead $\alpha = N \neq 0$, then the additional parameter $\xi \in \mathbb{C}$ would have strong consequences on the nature of the solutions. In fact, this parameter is responsible for symmetry-breaking phenomena; as we can obtain solutions of (2.2.8) which are no longer radially symmetric about any point.

The presence of non-radial solutions for this class of equations was first noticed by Chanillo–Kiessling in [CK1].

It is clear that other choices of f in (2.2.3) would yield to yet other classes of solutions for (2.2.8); see e.g., Section 5.5.5 in Chapter 5.

However, the examples above distinguish between two important classes of solutions—namely, those satisfying $e^u \in L^1(\mathbb{R}^2)$ as given by (2.2.9) and (2.2.11), and those with $e^u \notin L^1(\mathbb{R}^2)$ as given in (2.2.10).

In fact, as observed in [CW] for $\alpha = 0$, and subsequently in [PT] for $\alpha > 0$, the finite energy condition,

$$\int_{\mathbb{R}^2} e^u < +\infty, \tag{2.2.12}$$

only allows a function f of the "power-type" in (2.2.3). More precisely, the following classification result holds:

Theorem 2.2.1 *([CL1], [CW], [PT]) For $\alpha \geq 0$, every solution u of (2.2.8) satisfying (2.2.12) takes the form*

$$u(z) = \log \frac{8\lambda(\alpha+1)^2|z|^{2\alpha}}{\left(1 + \lambda|z^{\alpha+1} + \xi|^2\right)^2}, \tag{2.2.13}$$

with $\lambda > 0$, $\xi \in \mathbb{C}$, and $\xi = 0$ for $\alpha \notin \mathbb{Z}^+$. In particular,

$$\int_{\mathbb{R}^2} e^u = 8\pi(1+\alpha). \tag{2.2.14}$$

□

We emphasize that for $\alpha \notin \mathbb{Z}^+$, we must take $\xi = 0$ in (2.2.13); otherwise we would obtain a multivalued solution.

As already mentioned, such classification result is a consequence of Liouville formula (2.2.3) together with (2.2.5)–(2.2.7), and we refer to [CW] and [PT] for details. Let us mention that, the case $\alpha = 0$ was handled first by Chen–Li [CL1], by the method of "moving planes" of Alexandroff in the same spirit of [GNN] and [CGS]. Namely, in [CL1] the authors show that for $\alpha = 0$, all solutions of (2.2.8) and (2.2.12) are radially symmetric about a point, and consequently arrive at (2.2.13) (with $\alpha = 0$) after an O.D.E. analysis. Clearly, such an approach cannot work in general for the case $\alpha > 0$, since radial symmetry may be broken by the solutions of (2.2.8) when $\alpha \in \mathbb{N}$.

Nevertheless, the mere identity (2.2.14) can be derived from a general symmetry result obtained by Chen–Li [CL2] within the framework of the "moving plane" analysis, applied to v (the regular part of u) and defined as follows:

$$v(z) = u(z) - 2\alpha \log |z|. \tag{2.2.15}$$

Notice that v satisfies:

$$\begin{cases} -\Delta v = |z|^{2\alpha} e^{v} \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |z|^{2\alpha} e^{v} < +\infty. \end{cases} \tag{2.2.16}$$

In fact, for many purposes it is more convenient to analyze problem (2.2.16) rather than (2.2.8) and (2.2.12). In terms of (2.2.15), Theorem 2.2.1 reads as follows:

Corollary 2.2.2 *Let $\alpha \geq 0$, then every solution of (2.2.16) takes the form:*

$$v(z) = \log \frac{\lambda}{\left(1 + \lambda \gamma_\alpha |z^{\alpha+1} + \xi|^2\right)^2}, \; \gamma_\alpha = \frac{1}{8(1+\alpha)^2}, \tag{2.2.17}$$

with $\lambda > 0$, $\xi \in \mathbb{C}$, and $\xi = 0$ for $\alpha \notin \mathbb{Z}^+$. Furthermore,

$$\int_{\mathbb{R}^2} |z|^{2\alpha} e^{v} = 8\pi(1+\alpha). \tag{2.2.18}$$

Observe that, the scale invariance of (2.2.16) is expressed now according to the transformation:

$$v(z) \longrightarrow v_\mu(z) := v(\mu z) + 2(\alpha + 1) \log \mu, \; \forall \mu > 0.$$

We point out a symmetry property for the solutions of (2.2.16), which is not at all obvious from expression (2.2.17) when $\alpha \in \mathbb{N}$ and $\xi \in \mathbb{C} \setminus \{0\}$. To this purpose, for a given solution v of (2.2.16), we let

$$v_\infty = \lim_{|z| \to +\infty} \big(v(z) + 4(\alpha + 1) \log |z|\big). \tag{2.2.19}$$

By (2.2.17), the limit (2.2.19) exists and is finite.

Proposition 2.2.3 *Let $\alpha \geq 0$ and v be a solution of (2.2.16). Set*

$$\tau = \exp \frac{v(0) - v_\infty}{2(\alpha + 1)}$$

with v_∞ in (2.2.18). Then

$$v(z) = v\left(\frac{z}{\tau|z|^2}\right) + 2(\alpha+1)\log\frac{1}{\tau|z|^2}, \quad in\ \mathbb{R}^2.$$

Furthermore, setting $|z| = r$, we have

$$\begin{aligned} &\left(r - \frac{1}{\sqrt{\tau}}\right)(r\partial_r v + 2(\alpha+1)) < 0, \ if\, r \neq \frac{1}{\sqrt{\tau}} \\ &r\partial_r v + 2(\alpha+1) = 0, \ if\, r = \frac{1}{\sqrt{\tau}}. \end{aligned} \tag{2.2.20}$$

Properties (2.2.19) and (2.2.20) were established in a more general context in [PT, Theorem 2.5], to which we refer the reader for details.

2.3 Variational framework

As we shall see, many of the elliptic problems we shall analyze below admit a useful variational formulation, so that the search for their *solutions* can be reduced to finding the critical points of a Frechet differentiable functional,

$$I : E \longrightarrow \mathbb{R}$$

with E a suitable Banach space.

Recall that $v \in E$ is a *critical point* for I in E with a *critical value* $c = I(v)$, if we have $||I'(v)||_{E^*} = 0$. As usual, with E^* we denote the dual space of E, so that for a critical point v we have

$$\max_{\varphi \in E^*}(I'(v), \varphi) = 0. \tag{2.3.1}$$

Typically, the construction of a so-called Palais–Smale *(PS)-sequence*, namely a sequence $\{v_n\} \subset E$ such that

$$||I'(v_n)||_{E^*} \to 0, \quad \text{and } I(v_n) \to c \tag{2.3.2}$$

represents the first step in obtaining a critical point with a critical value c. For instance, if I is bounded from below, then (2.3.2) may be realized by a minimizing sequence and $c = \inf_E I$. See, for example [Gh].

Or, when I admits a “mountain-pass” structure (cf. [AR]), in the sense that

$$\begin{aligned} &\exists \Gamma \subset E \text{ a closed set which } \textit{separates} \text{ two points } v_0 \text{ and } v_1 \in E \text{ and} \\ &\inf_\Gamma I > \max\{I(v_0), I(v_1)\}. \end{aligned} \tag{2.3.3}$$

Then (2.3.2) may be realized via a min-max procedure, by considering the set of all continuous paths joining v_0 and v_1, (namely,

$$\mathcal{P} := \{\gamma : [0,1] \longrightarrow E \text{ continuous with } \gamma(0) = v_0 \text{ and } \gamma(1) = v_1\})$$

and by setting

$$c = \inf_{\gamma \in \mathcal{P}} \max_{t \in [0,1]} I(\gamma(t)) > \max\{I(v_0), I(v_1)\}. \tag{2.3.4}$$

In this case, the corresponding (PS)-sequence is obtained by using the flow associated to the pseudogradient vector field relative to I. More precisely,

Theorem 2.3.4 *Let $I \in C^1(E)$, and assume that there exist $c > 0$ and $\delta > 0$ such that*

$$\text{if } v \in E : \ |I(v) - c| < \delta \text{ then } \Rightarrow ||I'(v)||_{E^*} \geq \delta. \tag{2.3.5}$$

Then for any $\overline{\varepsilon} > 0$ sufficiently small and suitable $\varepsilon \in (0, \overline{\varepsilon})$, we find a deformation map

$$\eta : E \times [0,1] \longrightarrow E \tag{2.3.6}$$

satisfying:

$$\begin{aligned}
&\text{(i)} \quad \eta(v,0) = v \text{ and } I(\eta(v,t)) \text{ is decreasing in } t \in [0,1];\\
&\text{(ii)} \ \text{ if } v \in E : \ |I(v) - c| \geq \overline{\varepsilon}, \text{ then } \eta(v,t) = v \ \forall t \in [0,1];\\
&\text{(iii)} \text{ if } v \in E : \ I(v) < c + \varepsilon, \text{ then } I(\eta(v,1)) < c - \varepsilon.
\end{aligned} \tag{2.3.7}$$

We refer to [St1] and [R] for details and for more examples of min-max constructions that yield to critical points.

Once a (PS)-sequence for I (at level c) is available, in order to derive that c represents a critical value for I then we need to satisfy the following compactness condition:

Definition 2.3.5 *We say that the functional I satisfies the Palais–Smale condition (at level c), if any (PS)-sequence satisfying (2.3.2) admits a strongly convergent subsequence.*

Remark 2.3.6 (1) Any functional with a "mountain-pass" structure (in the sense of (2.3.3)) and satisfying the (PS) condition always admits a critical point with a relative critical value defined by (2.3.4), (cf. [AR]).

(2) By the direct method of the Calculus of Variations (cf. [Da]), we see that for a functional bounded from below, the (PS) condition at the level of the infimum may be replaced by (in fact is often equivalent to) the assumption of coerciveness and weakly lower semicontinuity.

2.4 Moser–Trudinger type inequalities

In our applications, a first situation of interest concerns the case where $E = H^1(\mathbb{R}^2)$, the Sobolev space obtained by the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$||v||_{H^1(\mathbb{R}^2)} = \left(||\nabla v||^2_{L^2(\mathbb{R}^2)} + ||v||^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}}, \tag{2.4.1}$$

and equipped with the scalar product,

$$(u, v) = \int_{\mathbb{R}^2} (\nabla u \nabla v + uv)\, dx\, dy.$$

By Sobolev's embedding theorem [Ada], we know that $\forall p \geq 2$:

$$H^1(\mathbb{R}^2) \hookrightarrow \mathrm{L}^p(\mathbb{R}^2),$$

and that the following inequality holds

$$||v||_{\mathrm{L}^p(\mathbb{R}^2)} \leq \pi \left(\frac{p-2}{p}\right)^{\frac{p-2}{2p}} ||v||_{H^1(\mathbb{R}^2)}, \ \forall v \in H^1\left(\mathbb{R}^2\right). \tag{2.4.2}$$

Inequality (2.4.2) was derived by Jaffe–Taubes (see Proposition VI.2.3 in [JT]) who aimed to determine an accurate (but not sharp) explicit dipendence on p of the constant involved in the Sobolev inequality, in order to deduce the following result:

Lemma 2.4.7 *There exist suitable constants $C > 0$ and $\gamma > 0$ such that, for every $v \in H^1(\mathbb{R}^2)$ there holds*

$$||e^v - 1||_{\mathrm{L}^2(\mathbb{R}^2)} \leq Ce^{\gamma ||v||^2_{H^1(\mathbb{R}^2)}}. \tag{2.4.3}$$

Moreover, for every $p \geq 2$ the map

$$\begin{aligned} H^1\left(\mathbb{R}^2\right) &\longrightarrow \mathrm{L}^p\left(\mathbb{R}^2\right) \\ v &\longrightarrow e^v - 1 \end{aligned} \tag{2.4.4}$$

is continuous. □

Proof. To establish (2.4.3), we use the Taylor expansion and write

$$\left(e^v - 1\right)^2 = 2\sum_{k=2}^{+\infty} \frac{2^{k-1}-1}{k!} v^k.$$

So, by means of (2.4.2) we can estimate:

$$\begin{aligned} \left|\left|e^v - 1\right|\right|^2_{\mathrm{L}^2(\mathbb{R}^2)} &\leq 2\sum_{k=2}^{+\infty} \frac{2^{k-1}-1}{k!} ||v||^k_{\mathrm{L}^k(\mathbb{R}^2)} \\ &\leq 2\sum_{k=2}^{+\infty} \frac{2^{k-1}-1}{k!} \left(\frac{k-2}{2}\right)^{\frac{k-2}{2}} ||v||^k_{H^1(\mathbb{R}^2)} \\ &\leq 2\sum_{k=2}^{+\infty} (2\pi)^k \left(\frac{k}{2}\right)^{\frac{k}{2}} \frac{1}{k!} ||v||^k_{H^1(\mathbb{R}^2)}. \end{aligned} \tag{2.4.5}$$

Since by Sterling's formula, we may find constants $c_2 > c_1 > 0$ such that

$$(c_1 k)^k \leq k! \leq (c_2 k)^k,$$

from (2.4.5), we readily arrive at (2.4.3). Thus, $e^v - 1 \in L^2(\mathbb{R}^2)$ and (2.4.4) follows for $p = 2$. On the other hand, for every $n \in \mathbb{N}$, we have

$$\left(e^v - 1\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} e^{kv} (-1)^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-1)^{n-k} \left(e^{kv} - 1\right) \in L^2\left(\mathbb{R}^2\right);$$

that is,

$$e^v - 1 \in L^{2n}(\mathbb{R}^2) \text{ and } \left\|e^v - 1\right\|_{L^{2n}(\mathbb{R}^2)} \leq C_n e^{\gamma_n \|\nabla v\|^2_{L^2(\mathbb{R}^2)}}$$

for every $n \in \mathbb{N}$, with $C_n > 0$ and γ_n suitable constants (depending on n). Therefore, by interpolation we conclude that (2.4.4) holds for any $p \geq 2$. □

The idea of using Sobolev's inequalities with explicit appropriate constants to estimate the norm of the exponential was adopted first by Trudinger in [Tr] in the framework of the Sobolev space $H_0^1(\Omega)$ with $\Omega \subset \mathbb{R}^2$ a bounded (regular) domain (see also [Sal]). Subsequently, Trudinger's result was re-derived by Moser in a more general form as follows:

Proposition 2.4.8 *([Tr], [Mo]) For $n \geq 2$ there exists a constant $C_n > 0$ (depending on n only) such that setting*

$$\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}} \tag{2.4.6}$$

with ω_n the volume of the unit sphere S^n in $\mathbb{R}^{n+1}$, we have

$$\sup\left\{\int_\Omega e^{\alpha_n |v|^{\frac{n}{n-1}}},\ v \in W_0^{1,n}(\Omega) \text{ and } \|\nabla v\|_{L^n(\Omega)} = 1\right\} \leq C_n|\Omega|, \tag{2.4.7}$$

where $|\Omega|$ is the Lebesgue measure of $\Omega \subset \mathbb{R}^n$.

Furthermore, α_n in (2.4.6) is the best possible constant for which the supremum in the left-hand side of (2.4.7) is finite.

Here, we are interested in considering (2.4.7) for the case $n = 2$, where it states that,

$$\forall v \in H_0^1(\Omega):\ \int_\Omega e^{4\pi\left(\frac{v}{\|\nabla v\|_{L^2(\Omega)}}\right)^2} \leq C|\Omega| \tag{2.4.8}$$

with a universal constant $C > 0$ and independent of $\Omega \subset \mathbb{R}^2$.

From (2.4.8) we easily derive the estimate

$$⨍_\Omega e^v \leq C e^{\frac{1}{16\pi}|\nabla v|^2_{L^2(\Omega)}},\ \forall v \in H_0^1(\Omega); \tag{2.4.9}$$

where we use standard notation to denote the average value of f as

$$\fint_{\Omega} f := \frac{1}{|\Omega|}\int_{\Omega} f. \tag{2.4.10}$$

The inequality (2.4.9) furnishes a "sharp" version of (2.4.3) for bounded domains. In fact, the value $\frac{1}{16\pi}$ is the best possible for inequality (2.4.9) to hold. This can be checked directly with the help of the "concentrating" solutions of the regular Liouville equation (i.e., (2.2.8) with $\alpha = 0$).

Indeed, by assuming for simplicity (and without loss of generality) that $\overline{B_1(0)} \subset \Omega$ and by setting

$$v_\lambda(z) = \begin{cases} \log\left(\frac{1+\lambda}{1+\lambda|z|^2}\right)^2 & \text{if } |z| \le 1 \\ 0 & \text{if } |z| \ge 1 \end{cases} \in H_0^1(\Omega), \tag{2.4.11}$$

we can check without great difficulty that, as $\lambda \to +\infty$,

$$||\nabla v_\lambda||^2_{L^2(\Omega)} = 16\pi \log(1+\lambda) + o(1) \tag{2.4.12}$$

and

$$\log \fint_{\Omega} e^{v_\lambda} = \log(1+\lambda) + O(1). \tag{2.4.13}$$

Therefore, as $\lambda \to +\infty$, v_λ furnishes an "optimal" family for (2.4.9) and shows that (2.4.9) does fail if we repace $\frac{1}{16\pi}$ with a *smaller* value.

The example above also indicates that it may not be always possible to obtain an extremal function for (2.4.9). Non-existence of an extremal function can be checked when Ω is a ball, while existence may occur for annulus or rectangular domains (see [Ban], [Su1], [CLMP1], and [CLMP2]). On the contrary, it is surprising to see that the existence of a minimizer can always be guaranteed for the original inequality (2.4.8), as it has been shown to in [CaCh], [Fl2], [Ln], and [St4].

It is of interest to us to consider possible versions of (2.4.8) and (2.4.9) over compact Riemann surfaces (M, g). We consider the case where M has no boundary (i.e., $\partial M = \emptyset$) and denote by ∇_g and $d\sigma_g$, respectively, the covariant derivative and the volume element induced by the Riemann metric g on M.

The inequality (2.4.8) continues to hold within the framework of the Sobolev space $H^1(M)$; that is

$$\sup\left\{\int_M e^{4\pi u^2}\, d\sigma_g, \quad u \in H^1(M) \text{ and } ||u||_{H^1(M)} = 1\right\} < +\infty \tag{2.4.14}$$

(see [Au], [Ad], [yLi1], and [yLi2] for generalizations). And as above, it yields to the inequality

$$\int_M e^u\, d\sigma_g \le C e^{\frac{1}{16\pi}\left(||\nabla_g u||^2_{L^2(M)} + ||u||^2_{L^2(M)}\right)}.$$

In particular, we see that for every $p \geq 1$, the map

$$\begin{aligned} H^1(M) &\longrightarrow L^p(\Omega) \\ u &\longrightarrow e^u \end{aligned} \tag{2.4.15}$$

is continuous.

On the other hand, on the subspace

$$E = \left\{ w \in H^1(M) : \int_M w \, d\sigma_g = 0 \right\}, \tag{2.4.16}$$

the norm $||w||_{H^1(M)}$ and $||\nabla_g w||_{L^2(M)}$ are equivalent, and it is possible to show that the exact same inequality as in (2.4.7) holds (see [Fo]). Namely,

Proposition 2.4.9 *([Fo]) Let M be a compact Riemann surface. There exists a constant $C > 0$ such that*

$$\fint_M e^w \, d\sigma_g \leq C e^{\frac{1}{16\pi} ||\nabla_g w||^2_{L^2(M)}}, \; \forall w \in E. \tag{2.4.17}$$

Interestingly enough, for the standard 2-sphere $M = S^2$, the inequality (2.4.17) can be derived as a limiting case (for $p \to +\infty$) of the well-known (sharp) Sobolev inequality

$$||u||^2_{L^p(S^2)} \leq \frac{p-2}{2\omega_2^{1-\frac{2}{p}}} ||\nabla u||^2_{L^2(S^2)} + \frac{1}{\omega_2^{1-\frac{2}{p}}} ||u||^2_{L^2(S^2)}, \; u \in H^1(S^2), \tag{2.4.18}$$

valid for $p \geq 2$ and $\omega_2 = |S^2| = 4\pi$ (see e.g., [Be] for a version of (2.4.18) in the dimension $n \geq 3$). Indeed, by applying (2.4.18) with $u = 1 + \frac{1}{p} w$ and $w \in E$, we find

$$\begin{aligned} \left(\int_{S^2} (1 + \frac{1}{p} w)^p \, d\sigma \right)^2 &\leq \left(\frac{p-2}{2\omega_2^{1-\frac{2}{p}} p^2} \int_{S^2} |\nabla w|^2 \, d\sigma + \omega_2^{\frac{2}{p}} + \frac{1}{p^2 \omega_2^{1-\frac{2}{p}}} \int_{S^2} w^2 \, d\sigma \right)^p \\ &= \omega_2^2 \left(1 + \frac{p-2}{2\omega_2 p^2} \int_{S^2} |\nabla w|^2 \, d\sigma + \frac{1}{\omega_2 p^2} \int_{S^2} w^2 \, d\sigma \right)^p. \end{aligned}$$

Hence, by passing to the limit as $p \to +\infty$, we derive the desired Moser–Trudinger inequality for functions defined over S^2:

$$\frac{1}{|S^2|} \int_{S^2} e^w \, d\sigma \leq e^{\frac{1}{16\pi} ||\nabla w||^2_{L^2(S^2)}}, \; \forall w \in H^1(S^2) : \int_{S^2} w \, d\sigma = 0 \tag{2.4.19}$$

(see [Mo], [On], [Au] and references therein).

The proof above is due to Beckner in [Be], which contains various other generalizations.

The derivation of (2.4.17) for general Riemann surfaces was obtained by Fontana in [Fo], along the lines of [Ad]. See also [DJLW1], [NT2], [ChCL], [Che], and [ChY3] for related results.

We may relate (2.4.19) to an inequality over $\mathbb{R}^2$ (to be compared with (2.4.3)) via the sterographic projection. To this purpose, consider S^2 embedded in $\mathbb{R}^3$, as given by the set of all $x \in \mathbb{R}^3 : \ ||x|| = 1$. Define the (inverse) stereographic projection with respect to the south pole $(0, 0, -1)$ as

$$\begin{aligned} \pi : \mathbb{R}^2 &\longrightarrow S^2 \setminus \{(0,0,-1)\} \\ z = (x, y) &\longrightarrow (\rho z, t) \end{aligned}$$

with $\rho = \frac{2}{1+|z|^2}$ and $t = \frac{1-|z|^2}{1+|z|^2}$.

Thus, for every $v \in H^1(S^2)$, we may consider a function $u = v \circ \pi$ defined over $\mathbb{R}^2$ and verify that:

$$\begin{aligned} \int_{S^2} |v|^p \, d\sigma &= \int_{\mathbb{R}^2} \left(\frac{2}{1+|z|^2}\right)^2 |u|^p \, dx\, dy, \ \forall p \geq 1; \\ \int_{S^2} e^v \, d\sigma &= \int_{\mathbb{R}^2} \left(\frac{2}{1+|z|^2}\right)^2 e^u \, dx\, dy; \\ \int_{S^2} |\nabla v|^2 \, d\sigma &= \int_{\mathbb{R}^2} |\nabla u|^2 \, dx\, dy. \end{aligned} \tag{2.4.20}$$

Therefore, if we set

$$E = \left\{ u : \ |\nabla u| \in \mathrm{L}^2\left(\mathbb{R}^2\right) \quad \text{and} \quad \frac{u}{1+|z|^2} \in \mathrm{L}^2\left(\mathbb{R}^2\right) \right\}, \tag{2.4.21}$$

then

$$\begin{aligned} &v \in H^1(S^2) \text{ if and only if } u = v \circ \pi \in E, \\ &\int_{S^2} v \, d\sigma = 0 \text{ if and only if } \int_{\mathbb{R}^2} \frac{u}{\left(1+|z|^2\right)^2} \, dx\, dy = 0, \end{aligned}$$

and (2.4.19) reduces to

$$\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{e^u}{\left(1+|z|^2\right)^2} \leq \exp\left(\frac{1}{16\pi} ||\nabla u||^2_{\mathrm{L}^2(\mathbb{R}^2)} + \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{u}{\left(1+|z|^2\right)^2} \right). \tag{2.4.22}$$

These observations permit us to use the classification result (2.2.17) with $\alpha = 0$, to conclude the following result first noticed by Onofri in [On]:

Proposition 2.4.10 *([On], [Ho]) The equality in (2.4.19) is attained if and only if* $w = 0$.

Proof. Assume that w attains the equality in (2.4.19). Without loss of generality, we may suppose that $\max_{S^2} w$ is attained at the north pole $(0, 0, 1)$. Therefore, the function $u = (w \circ \pi - \log \frac{1}{4\pi} \int_{S^2} e^w \, d\sigma) \in E$ attains equality in (2.4.22) and satisfies

$$\begin{cases} -\Delta u = \dfrac{8}{\left(1+|z|^2\right)^2}(e^u - 1) \text{ in } \mathbb{R}^2 \\ u(0) = \max\limits_{\mathbb{R}^2} u, \ \displaystyle\int_{\mathbb{R}^2} \dfrac{e^u}{\left(1+|z|^2\right)^2} = \pi. \end{cases}$$

Consequently, setting

$$U(z) = \log \frac{8}{\left(1+|z|^2\right)^2} + u(z),$$

we see that it satisfies

$$\begin{cases} -\Delta U = e^U \text{ in } \mathbb{R}^2 \\ U(0) = \max\limits_{\mathbb{R}^2} U, \ \displaystyle\int_{\mathbb{R}^2} e^U = 8\pi. \end{cases}$$

Thus, we can use Corollary 2.2.2, with $\alpha = 0$, to conclude that by necessity:

$$U(z) = \log \frac{8\lambda}{(1+\lambda|z|^2)^2}, \quad \text{for some } \lambda > 0.$$

In other words,

$$u(z) = \log \lambda + 2 \log \frac{1+|z|^2}{1+\lambda|z|^2}.$$

Since $|\nabla u| \in L^2(\mathbb{R}^2)$ $\lambda = 1$ must hold, which implies $u = 0$; and so $w = 0$ as claimed. □

Analogous inequalities over $\mathbb{R}^2$ have been derived by McOwen [McO] with other weight functions. We also refer to [DET] for more general weighted inequalities of the (2.4.22) type, and their influence in the symmetry-breaking phenomena of the Hardy-Sobolev inequality (cf. [CKN]).

Versions of the above mentioned inequalities are available also for the case when M admits a non-empty boundary $\partial M \neq \emptyset$ (see [Fo] and [Au] and references therein). However, in this case, the constant $\frac{1}{16\pi}$ is no longer appropriate. We know that for a smooth ∂M, the constant must be replaced by $\frac{1}{8\pi}$, while its value becomes more involved when ∂M admits corners, as it has been discussed in [ChY1].

For manifolds other than the 2-sphere, the existence of a minimum for the extremal problem (2.4.14) can always be ensured (see [yLi1] and [yLi2]).

However, we encounter a delicate problem when investigating the possibility of attaining equality in (2.4.17). Of particular interest to us is the case of the flat 2-torus $M = \mathbb{R}^2 \diagup \mathbf{a}_1\mathbb{Z} \times \mathbf{a}_2\mathbb{Z}$ where $\mathbf{a}_1$ and $\mathbf{a}_2$ are two linearly independent vectors of $\mathbb{R}^2$ that generate the periodic cell domain $\Omega = \{s\mathbf{a}_1 + t\mathbf{a}_2,\ 0 < s,\ t < 1\}$ in (2.1.27). Since

$$H^1\left(\mathbb{R}^2 \diagup \mathbf{a}_1\mathbb{Z} \times \mathbf{a}_2\mathbb{Z}\right) = \left\{v \in H^1_{\text{loc}}(\mathbb{R}^2) : v \text{ doubly periodic relatively to the cell domain } \Omega\right\} := \mathcal{H}(\Omega), \tag{2.4.23}$$

the Moser–Trudinger inequality (2.4.17) can be expressed in terms of doubly periodic functions as

$$\fint_{\Omega} e^{w} \leq C e^{\frac{1}{16\pi}||\nabla w||^2_{L^2(\Omega)}}, \ \forall w \in \mathcal{H}(\Omega) : \int_{\Omega} w = 0, \tag{2.4.24}$$

with a suitable constant $C > 0$.

Sticking to our complex notations, we are going to identify the (independent) vectors $\mathbf{a}_j = \alpha_j \mathbf{e}_1 + \beta_j \mathbf{e}_2$ with the complex number $w_j = \alpha_j + i\beta_j, \ j = 1, 2$, so that $\langle i w_1, w_2 \rangle = \mathcal{R}e(\, i w_1 \overline{w_2}) \neq 0$ formulates the condition for linear independency. In this way, we may express the flat torus as $M = \mathbb{C} \diagup (w_1\mathbb{Z} + w_2\mathbb{Z})$.

By the work in [DJLW1] and [NT2], we know that for $M = \mathbb{C} \diagup (a\mathbb{Z} + ib\mathbb{Z})$ and $0 < a \leq b$, the equality in (2.4.17) (or (2.4.24)) can always be attained. Actually, the corresponding extremal function is *not* identically equal to zero when $\frac{a}{b} < \frac{2}{\pi}$. Other interesting results related to the study of extremals for the Moser–Trudinger inequality can be found in [CLS], [ChCL], [ChLW], [LiL], [LiL1], [LiW], and [LiW1].

2.5 A first encounter with mean field equations of Liouville-type

For a given non-negative function $h \in L^{\infty}(M)$ and $\mu > 0$, we may consider the functional

$$I_{\mu}(w) = \frac{1}{2}||\nabla_g w||^2_{L^2(M)} - \mu \log \int_M h e^{w}\, d\sigma_g, \ w \in E \tag{2.5.1}$$

(recall E in (2.4.16)).

By virtue of (2.4.15), the functional $I_{\mu} \in C^1(E)$, and its critical points correspond to (weak) solutions for the *mean field equation* of the Liouvill-type

$$\begin{cases} -\Delta_g w = \mu \left(\frac{he^{w}}{\int_M he^{w}\, d\sigma_g} - \frac{1}{|M|} \right) \text{ in } M \\ \int_M w\, d\sigma_g = 0 \end{cases} \tag{2.5.2}$$

with Δ_g the Laplace–Beltrami operator corresponding to the Riemannian metric g on M, and $|M|$ the surface area of M.

We shall be interested in handling (2.5.1) or (2.5.2) under the following set of assumptions on h:

$$h = e^{u_0} \in L^{\infty}(M) : \ u_0 \in L^1(M) \text{ and } \int_M u_0\, d\sigma_g = 0. \tag{2.5.3}$$

Notice that the last condition in (2.5.3) implies no real restriction on h, since problem (2.5.2) remains unchanged if we replace h with th, $t > 0$.

Equation (2.5.2) has attracted much attention in the last two decades by the central role it has played in a variety of problems arising in conformal geometry (see e.g., [Au], [Ba], [ChY1], [ChY2], [ChY3], [CL], [CD], [CL3], [CK3], [H], [Ho], [K], [KW1], [KW2], [L1], [Ob], [Ni], [On], and the references therein), mathematical physics (see e.g., [CLMP1], [CLMP2], [CK1], [CK2], [CK3], [On], [Ki1], [Ki2], [Wo], and the references therein) and applied mathematics (e.g., [Cha], [Ci], [CP], [BE], [Ge], [KS], [EN], and [Mu]). In our context, problem (2.5.2) has entered in a crucial way in the understanding of the asymptotic behavior of "non-topological" Chern–Simons vortices and in the study of electroweak mixed states.

The solvability of (2.5.2) poses a rather delicate problem, as we can see already on the basis of (2.4.24). This leads us to distinguish the following cases:

Case 1: $\mu \in (0, 8\pi)$, then the functional I_μ is coercive and bounded from below in E. Since it is also weakly lower semicontinuous, it attains its infimum at a solution of (2.5.2). Hence, problem (2.5.2) is always solvable in this case. Actually, for the flat 2-torus, where (2.5.2) reduces to a periodic boundary value problem, it is possible to use the Weierstrass $\mathcal{P}$-function into the Liouville formula (2.2.3) to exhibit an explicit solution when $\mu = 4\pi$. For details see [Ol].

In this case, it is important to understand under which circumstances we can claim *uniqueness* of the solution (or of the minimizer).

When h is a constant, this amounts to asking if problem (2.5.2), with $\mu \in (0, 8\pi)$, admits only the trivial solution $w = 0$. We know the answer to be affirmative for the 2-sphere (cf. [On], [Ho], [CK1], and [Li1]); whereas multiplicity does occur for the flat 2-torus, $M = \mathbb{C} / (a\mathbb{Z} + ib\mathbb{Z})$ and $0 < a \leq b$, provided that $\frac{a}{b} < \frac{2}{\pi}$ and $4\pi^2 \frac{a}{b} < \mu < 8\pi$ (see [DJLW1], [NT2], and [CLS]). On the other hand, a result of Cabré–Lucia–Sanchon in [CLS], establishes uniqueness for the case $0 < \mu \leq \min\{4\pi^2 \frac{a}{b}, \lambda_*\}$, with $\lambda_* = \lambda_*(\frac{a}{b}) < 8\pi$ an explicit constant depending on the conformal radius of the periodic cell domain. More precisely, the authors in [CLS] show that for $0 < \mu < \lambda_*$ the solutions must be constant with respect to one variable. Then they can conclude, by the O.D.E. analysis of [RT2], that one-dimensional non-trivial solutions do not exist for μ below the value $4\pi^2 \frac{a}{b} = \lambda_1 |\Omega|$, where λ_1 is the first non-zero eigenvalue for $-\Delta$ in $\mathcal{H}(\Omega)$. In particular, in [CLS] the authors are able to improve upon the analysis of [RT2] for the case $\frac{a}{b} \leq \frac{1}{2}$; they show that $u = 0$ is the only solution (not necessarily one-dimensional) for (2.5.2) when h is a constant, if and only if $0 < \mu < 4\pi^2 \frac{a}{b}$.

By a different argument based on the isoperimetric profile of M, Lin–Lucia in [LiL] were able to obtain a uniqueness result which is sharp when $\frac{a}{b} \geq \frac{\pi}{4}$, since it holds for the full range of parameters $\mu \in [0, 8\pi]$. The approach of [LiL] permits also to improve some other results in [CLS].

Actually, the best available result about uniqueness, for any value $\mu \in [0, 8\pi]$, concerns the global minimizer of I_μ, as obtained recently in [LiL1]. This result can be viewed as the equivalent on the flat 2-torus of Onofri's result (cf. [On]) on the 2-sphere. See also [LZ] for related results.

Case 2: If $\mu = 8\pi$, then $I_{\mu=8\pi}$ is bounded from below but is no longer coercieve in E, and $I_{\mu=8\pi}$ fails to satisfy the (PS)-condition. In fact, the existence of a minimizer for $I_{\mu=8\pi}$ is influenced greatly by the nature of the given weight function h.

For instance, when $M = S^2$, a minimizer for $I_{\mu=8\pi}$ exists if and only if h is a constant function (see [Ho]).

If h is not constant for $M = S^2$, then more elaborate min-max procedures must be introduced in order to obtain solutions to (2.5.2) when $\mu = 8\pi$ (see e.g., [Mo], [ChY1], [ChY2], [ChY3], [L1], [Li2], and the references therein). These existence results are of particular interest because of their connection to the assigned Gauss curvature problem, see [KW1] and [KW2] for details.

For the flat 2-torus $M = \mathbb{C}/(a\mathbb{Z} + ib\mathbb{Z})$, it is shown in [NT2] that the functional $I_{\mu=8\pi}$ always attains its infimum on E provided h satisfies the following condition:

$$h = e^{u_0} \in C(M), \exists q \in M : \; u_0(q) = \max_M u_0 \text{ and } -\Delta u_0(q) < \frac{8\pi}{|M|}. \quad (2.5.4)$$

We shall give a detailed proof of this result in Section 6.3 of Chapter 6, and refer to [DJLW1] and [ChL2] for analogous results over a general surface, where (2.5.4) is replaced by a similar condition involving the Gauss curvature.

For our periodic Chern–Simons vortex problem (2.1.12), equation (2.5.2) occurs with $\mu = 4\pi N$ and u_0 in (2.5.3) given by the *unique* solution for the problem

$$\begin{cases} \Delta_g u_0 = 4\pi \sum_{j=1}^{N} \delta_{z_j} - \frac{4\pi N}{|M|} \text{ in } M, \\ \int_M u_0 \, d\sigma_g = 0 \end{cases} \quad (2.5.5)$$

with $\{z_1, \ldots, z_N\} \subset M$ the assigned set of vortex points, repeated according to their multiplicity. Note that u_0 in (2.5.5), attains its maximum value at a point $q \in M \setminus \{z_1, \ldots, z_N\}$, where we have: $-\Delta u_0(q) = \frac{4\pi N}{|M|}$. Since $\mu = 8\pi$ in (2.5.2) occurs exactly when $N = 2$ (i.e., in the *double* vortex case), we see that condition (2.5.4) is just violated by u_0 in (2.5.5) in this case. Thus, the above mentioned result does not apply, and in fact, Chen–Lin–Wang in [ChLW] were able to show that when the two vortex points coincide, $I_{\mu=8\pi}$ *cannot* attain its infimum in E. Moreover, in this situation, namely when $h = e^{u_0}$ with u_0 satisfying (2.5.5) with $N = 2$ and $z_1 = z_2$, the authors in [ChLW] provide uniform estimates for the solutions of the corresponding equation (2.5.2) with $\mu = 8\pi$. This allows us to define the Leray–Schauder degree at zero for the associated Fredholm map $F = Id + 8\pi T_h : E \to E$,

$$T_h w = \Delta_g^{-1} \left(\frac{h e^w}{\int_M h e^w d\sigma_g} - \frac{1}{|M|} \right) \in E, \; w \in E, \quad (2.5.6)$$

where T_h defines a compact operator and $\Delta_g^{-1} : E \to E$ defines the inverse operator of Δ_g acting on E. In [ChLW], this degree is computed to be equal to zero. This suggests that the given problem, which we can formulate in terms of $u = u_0 + w$, as follows:

$$\begin{cases} -\Delta_g u = 8\pi \left(\frac{e^u}{\int_M e^u d\sigma_g} - \delta_{z_1} \right) \text{ in } M \\ \int_M u \, d\sigma_g = 0 \end{cases} \quad (2.5.7)$$

with $z_1 \in M$, admits no solutions when $M = \mathbb{C}/(a\mathbb{Z} + ib\mathbb{Z})$. This fact has been subsequently proved by Lin–Wang in [LiW] and [LiW1] by a clever use of the Weierstrass $\mathcal{P}$-function, which naturally enters in the *Liouville formula* (2.2.3) in order to fulfil the periodic boundary conditions. More generally, the authors in [LiW] are able to relate the existence of a solution for (2.5.7) on $M = \mathbb{C}/(w_1\mathbb{Z} + w_2\mathbb{Z})$ to the number of critical points of the Green's function G, in the fundamental domain $\Omega = \left\{z = tw_1 + sw_2,\ s,t \in \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}$ (see Remark 2.5.11 below). Since the critical point's equation for G can be formulated in terms of $\mathcal{P}$, the authors analyze this latter function in order to arrive at their non-existence result, (see [LiW] for details). Actually, the study carried out by Lin–Wang in [LiW] and [LiW1] permits to conclude that (2.5.7) can admit *at most* a solution $u = u_0 + w$ with the regular part w corresponding to a minimum for $I_{\mu=8\pi}$ in E. This fact implies, in particular, that G admits at most five critical points — a property interesting in its own right. Moreover, if G admits exactly three critical points in Ω (i.e., the three half-periods $\frac{w_1}{2}$, $\frac{w_2}{2}$ and $\frac{1}{2}(w_1 + w_2)$ that occur for the *rectangle* where $w_1 = a$ and $w_2 = ib$) then non-existence holds; while if G admits additional critical points (as occurs for the *rhombus* with $w_1 = a$ and $w_2 = \frac{a}{2}(1 + i\sqrt{3})$), then existence holds. The authors in [LiW1] also deduce information on the nature of these additional critical points.

Case 3: If $\mu > 8\pi$, then the functional I_μ is unbounded in E, and min-max critical values must be sought in order to obtain solutions for (2.5.2). Following [ST], [DJLW3] and [BT2], we shall give an example of such min-max construction in Section 6.2 of Chapter 6. Notice that in this case we face the difficulty of checking that the (PS)-condition holds for I_μ. To this end, we use Struwe's monotonicity trick (cf. [St1], [St2], [St3], and [Je]), as we see that I_μ is decreasing with respect to the parameter μ. Indeed, recalling Jensen's inequality,

$$\text{if } F : \mathbb{R} \longrightarrow \mathbb{R} \text{ is convex then } F\left(\fint_M u\, d\sigma_g\right) \le \fint_M F(u)\, d\sigma_g, \tag{2.5.8}$$

and using it with $F(t) = e^t$, we find that

$$\fint_M e^{u_0+w}\, d\sigma_g \ge 1,\ \forall w \in E \text{ and } \int_M u_0\, d\sigma_g = 0. \tag{2.5.9}$$

Thus, in view of (2.5.3), for $\mu_1 \le \mu_2$, we have $I_{\mu_1}(w) \ge I_{\mu_2}(w),\ \forall w \in E$.

We obtain an existence result for (2.5.2), when M admits *genus* $g > 0$, $\mu \in (8\pi, 16\pi)$ and (2.5.3) holds. In particular we are allowed to take u_0 as in (2.5.5). See [ST], [DJLW3], [BT2], and Section 6.2 in Chapter 6 for a detailed proof.

We mention that, when the value of the parameter $\mu \ge 16\pi$ and (beside (2.5.3)), we assume that $h > 0$ in M, then existence results for (2.5.2) can be deduced by the degree formula obtained by Chen–Lin in [CL1] and [CL2]. More precisely, if $h \in C^{0,1}(M)$ is *strictly* positive on M, then Li in [L2] showed that, for every $\mu \in \mathbb{R}^+ \backslash 8\pi N$, the Leray-Schauder degree d_μ at zero of the Fredholm map $Id + \mu T_h$, with T_h in

(2.5.6), is well-defined. Moreover, for $\mu \in (8\pi(n-1), 8\pi n)$ the degree depends only on the integer $n \in \mathbb{N}$, and on the topological properties of M.

Subsequently, Chen–Lin in [ChL1] and [ChL2] were able to complete Li's analysis and arrived at the following formula:

$$d_\mu = \begin{cases} 1, & \text{if } \mu \in (0, 8\pi) \\ \frac{(-\chi(M)+1)\dots(-\chi(M)+n-1)}{(n-1)!}, & \text{if } \mu \in (8\pi(n-1), 8\pi n),\ n \in \mathbb{N} \setminus \{1\} \end{cases} \tag{2.5.10}$$

where $\chi(M) = 2(1-g)$ is the Euler characteristic of M with genus g.

Notice that for any manifold M with *positive genus*, in particular the flat 2-torus, we see that $d_\mu > 0,\ \forall \mu \in \mathbb{R}^+ \setminus 8\pi\mathbb{N}$, and so we can ensure the existence of a solution for (2.5.2) in this case.

More precisely in [ChL2] it is shown that for the flat 2-torus we have $d_\mu = 1, \forall \mu \in (0, +\infty)$. However, this does not imply that the corresponding solutions are uniformly bounded, see [LiL], [LiW], and [Lu]. For the standard sphere $M = S^2$, we have $d_\mu = -1$ for $\mu \in (8\pi, 16\pi)$, and so existence is guaranteed in this case; while $d_\mu = 0$ for any larger value $\mu \notin 8\pi\mathbb{N}$. This leaves the question of existence as a challenging, open problem in this case. See [Dj] for some contribution in this direction.

Again we stress that the results above *do not* apply when $h = e^{u_0}$ and u_0 is given by (2.5.5). Indeed, in this case,

$$u_0(z) = 4\pi \sum_{j=1}^{N} G(z, z_j)$$

where $G(z, p)$ defines the Green's function of Δ_g in $H^1(M)$, satisfying:

$$\begin{cases} \Delta_g G(\cdot, p) = \delta_p - \frac{1}{|M|} \text{ in } M, \\ \int_M G(\cdot, p)\, d\sigma_g = 0. \end{cases} \tag{2.5.11}$$

Note that $G(z, p) = G(p, z)$ and as is well-known (cf. [Au]),

$$G(z, p) = \frac{1}{2\pi} \log(d_g(z, p)) + \gamma(z, p),$$

where $d_g(\cdot, \cdot)$ denotes the distance function on M, and γ (the regular part of G) is a suitable smooth function defined on $M \times M$.

Consequently,

$$h(z) = e^{u_0(z)} = \prod_{j=1}^{N} d_g^2(z, z_j) V(z), \tag{2.5.12}$$

with suitable

$$0 < V \in C^{0,1}(M).$$

Thus, we see that h *vanishes* exactly at z_j with order $2n_j$, where n_j is the multiplicity of z_j, $j = 1, \dots, N$.

Remark 2.5.11 For later use, we recall that for the flat 2-torus $M = \mathbb{C}/(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$, the Green's function is identified by a doubly periodic function over the periodic cell domain $\Omega = \left\{z = s\omega_1 + t\omega_2 : 0 < |s|, |t| < \frac{1}{2}\right\}$. In fact, $G(z, y) = G(z - y)$ and $\int_\Omega G = 0$. Furthermore, G is even in Ω and can be expressed in terms of the Theta function θ_1 as

$$G(z) = -\frac{1}{2\pi}\log\frac{|\theta_1(z)|}{|\theta_1'(0)|} + \frac{y^2}{2\beta}, \tag{2.5.13}$$

where $z = x + iy$, $\tau = \frac{\omega_1}{\omega_2} = \alpha + i\beta$ and $\theta_1(z) = -i\sum_{n=-\infty}^{+\infty}(-1)^n e^{i\pi((2n+1)z+(n+\frac{1}{2})^2\tau)}$. See [LiW] for details.

For a rectangle, where $\omega_1 = a$ and $\omega_2 = ib$, we can carry out explicit calculations and thus deduce that G takes the form

$$G(z) = \frac{1}{2\pi}\log|z| + \gamma(z), \tag{2.5.14}$$

with γ (smooth in $\overline{\Omega}$) expressed as

$$\begin{aligned}\gamma(z) = \gamma(x_1 + ix_2) = &\sum_{(m,n)\neq(0,0)} \varepsilon_n\varepsilon_m \cos(\lambda_m x_1)\cos(\mu_n x_2)\frac{e^{-(\lambda_m^2+\mu_n^2)}}{\lambda_m^2 + \mu_n^2}\\ &+ \int_0^1 \frac{ds}{4\pi s}\sum_{(m,n)\neq(0,0)}\sum_{-\infty}^{+\infty}\exp\left(-\frac{(x_1 - ma)^2 + (x_2 - nb)^2}{4s}\right)\\ &- \frac{1}{4\pi}\int_{\frac{|x|^2}{4}}^1 \frac{dt}{t}(1 - e^{-t}) + \frac{1}{4\pi}\log 4 + \frac{1}{4\pi}\int_1^{+\infty}\frac{e^{-t}}{t}\,dt - 1, \end{aligned}\tag{2.5.15}$$

where $\lambda_m = \frac{2\pi}{a}m$, $\mu_n = \frac{2\pi}{b}n$, and $\varepsilon_n = 1$ or 2 according to whether $n = 0$ or $n \geq 1$, respectively (cf. [Tit]).

If we take h as in (2.5.12), it is still possible to show that d_μ is well-defined for any $\mu \in \mathbb{R}^+ \setminus 8\pi\mathbb{N}$ (see [BT2], [T4], and [T5]). Furthermore, $d_\mu = 1$ for $\mu \in (0, 8\pi)$ and $d_\mu = \chi(M) + N + 1$ for $\mu \in (8\pi, 16\pi)$, provided the zeroes of h in M are all *distinct*, that is, in (2.5.12) we have $z_j \neq z_k$ for $j \neq k \in \{1, \dots N\}$, (see [ChLW]). Thus, when $h > 0$ in M, formally we can take $N = 0$ in the above formula and reduce to (2.5.10). However, when h vanishes as in (2.5.12) and $\mu \in (16\pi, +\infty) \setminus 8\pi\mathbb{N}$, then a general formula for d_μ is not yet available.

We expect that the knowledge of the expression for d_μ would carry relevant information about the N-vortex problem. This we see already when M is the flat 2-torus, where we can use the above formula to see that d_μ admits a jump when crossing the value $\mu = 8\pi$, from $d_\mu = 1$ (for $\mu < 8\pi$) to $d_\mu = N + 1$ (for $8\pi < \mu < 16\pi$).

On the contrary, when $N = 0$ then $d_\mu = 1$ for any $\mu \in (0, 16\pi) \setminus \{8\pi\}$.

Other contributions related to elliptic problems involving exponential nonlinearities can be found for instance in [Ban], [BP], [Ch2], [DeKM], [DDeM], [Es], [EGP], [Ge], [MW], [M], [NS], [Ni], [P], [Sp], [Su1], [Su2], [WW1], [WW2], [Wes], [Wo], and [Y8]. The analysis of PS-sequences for I_μ can be found in [OS1].

Extensions of the results discussed above to the case of systems are of much interest in applications (see e.g., [CSW], [SW1], [SW2]). More precisely, for an assigned symmetric invertible positive definite $n \times n$ matrix $A = (a_{ij})_{i,j=1,\dots,n}$ we are interested in analyzing the functional

$$I_\mu(w_1, \dots, w_n) = \frac{1}{2}\int_M \sum_{i,j=1}^{n} a^{i,j} \nabla w_i \nabla w_j \, d\sigma_g - \sum_{j=1}^{n} \mu_j \log\left(\fint_M h_j e^{w_j} \right) d\sigma_g, \tag{2.5.16}$$

where $A^{-1} = (a^{ij})_{i,j=1,\dots,n}$ denotes the inverse of the given matrix A, $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{R}^+)^n$ is a n-ple of assigned positive numbers, $h_j \in L^\infty(M)$ is a given non-negative function, and $w_j \in E$, for every $j = 1, \dots, n$.

We shall naturally arrive at considering a functional of the type (2.5.16) in our study of non-abelian $SU(n+1)$-vortices with matrix $A = K$ as given by the Cartan matrix in (1.3.79) relative to the gauge group $SU(n+1)$. For later purposes, recall that the Cartan matrix $K = (K_{ij})_{i,j=1,\dots,n}$ is identified by the condition:

$$K_{ij} = 2\delta^i_j - \delta^{i+1}_j - \delta^i_{j+1}. \tag{2.5.17}$$

The problem to determine sharp conditions on the n-ple $\mu = (\mu_1, \dots, \mu_n)$, such that I_μ is bounded from below in E^n was first treated by Chipot–Shafrir–Wolansky in [CSW], who actually considered the Dirichlet analog of the functional (2.5.16) as described in (2.5.19) below. In [CSW], the authors assume that the entries of the matrix A are non-negative, a condition justified by their aim to treat models in population dynamics in absence of conflicts.

They introduce $2^n - 1$ quadratic polynomials $\Lambda_J(\mu)$ defined for every non-empty subset $J \subset \{1, \dots, n\}$ as

$$\Lambda_J(\mu) = \sum_{k \in J} \mu_k \left(8\pi - \sum_{j \in J} a_{kj} \mu_j \right), \tag{2.5.18}$$

and prove that, for a regular domain $\Omega \subset \mathbb{R}^2$ the Dirichlet functional (to be compared with (2.5.16)),

$$J_\mu(w_1, \dots, w_n) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a^{i,j} \nabla w_i \nabla w_j - \sum_{j=1}^{n} \mu_i \left(\log \fint_\Omega h_j e^{w_j} - \int_\Omega w_j \right), \tag{2.5.19}$$

is bounded from below in $(H^1_0(\Omega))^n$ provided

$$\Lambda_J(\mu) > 0, \ \forall J \subseteq \{1, \dots, n\}. \tag{2.5.20}$$

On the contrary, such a boundeness property is violated when the opposite inequality holds in (2.5.20), for some $J \subseteq \{1, \dots, n\}$.

This result was extended by Wang in [W] for the functional I_μ in (2.5.16). Since (2.5.20) reduces exactly to the Moser–Trudinger condition for $n = 1$ (see (2.4.17)), it is natural to ask whether I_μ (or J_μ) remains bounded from below in E^n (or in $(H_0^1(\Omega))^n$) even when we allow equality in (2.5.20). We thus relax to the condition:

$$\Lambda_J(\mu) \geq 0, \ \forall J \subseteq \{1, \dots, n\}. \tag{2.5.21}$$

In this respect, an affirmative answer was given by Wang [W] for stochastic matrices $A = (a_{ij})_{i,j=1,\dots,n}$ satisfying:

$$a_{ij} \geq 0, \ \forall i, j = 1, \dots, n \text{ and } \sum_{j=1}^{n} a_{ij} = 1, \ \forall i \in \{1, \dots, n\}. \tag{2.5.22}$$

Wang's approach was pursued further by Jost–Wang in [JoW1] for the Toda system of interest here, namely, when we take $A = K$, the Cartan matrix relative to $SU(n+1)$ given in (2.5.17). Notice that, in this case, (2.5.21) reduces to the condition:

$$\mu_j \leq 4\pi, \ \forall j = 1, \dots, n. \tag{2.5.23}$$

We summarize such results in the following:

Theorem 2.5.12 *([W], [JW]) Under the assumption (2.5.22) or when $A = K$, the condition (2.5.21), or respectively (2.5.23), is necessary and sufficient for I_μ to be bounded from below in E^n.*

Moreover, when $A = K$ and equality holds in (2.5.23) for some $j \in \{1, \dots, n\}$, the existence of a minimizers for I_μ has been established by Jost–Lin–Wang in [JoLW], under a condition analogous to (2.5.4). Whereas, when (2.5.23) is violated and so I_μ is no longer bounded from below, the existence of a critical point for I_μ is established in [LN], [ChOS], [MN], and [JoLW], in the same spirit of [ST] and [DJLW1]. We mention that in [JoLW] one can also find a degree formula for the corresponding $SU(n+1)$-Toda system valid for a certain range of parameters $\mu_j, \ j = 1, \dots, n$.

Returning to a general matrix $A = (a_{ij})_{i,j=1,\dots,n}$ we mention the work of Shafrir–Wolansky [SW1], concerning the functional I_μ over $M = S^2$. The authors in [SW1] are able to identify, in (2.5.10), the necessary and sufficient condition such that I_μ (with $M = S^2$) is bounded from below in E^n, provided some mild condition holds for the entries of the matrix A. We refer to [SW1] for details; here we merely mention that the approach of Shafrir–Wolansky relies on a "duality" method, which in particular yields to a simple and direct proof of the results in [W] and [JoW1]. See [SW1] also for a general discussion on the properties of the functional I_μ in relation to the properties of the matrix A.

The Euler–Lagrange equation relative to the functional (2.5.16) leads to the following system of mean field equations of the Liouville-type in the variable $(w_1, \dots, w_n)$:

$$\begin{cases} -\Delta w_i = \displaystyle\sum_{j=1}^{n} a_{ij}\mu_j \left(\frac{h_j e^{w_j}}{\int_M h_j e^{w_j}\, d\sigma_g} - \frac{1}{|M|} \right) \text{ in } M, \\ \displaystyle\int_M w_i\, d\sigma_g = 0, \ i = 1, \dots, n. \end{cases} \tag{2.5.24}$$

For problem (2.5.24), one may try to establish properties similar to those discussed above for the single equation (2.5.2). However, aside from the coercitivity condition, (2.5.20), very little is known about the structure of the solution set of (2.5.24). Interesting progress has been obtained recently for the case when we take $a_{ij} = K_{ij}$ in (2.5.17); namely, A coincides with the Cartan matrix K relative to $SU(n+1)$.

In this case, (2.5.24) takes the structure of the following Toda system:

$$\begin{cases} -\Delta w_i = \displaystyle\sum_{j=1}^{n} \left(2\delta^i_j - \delta^{i+1}_j - \delta^i_{j+1} \right) \mu_j \left(\frac{h_j e^{w_j}}{\int_M h_j e^{w_j}\, d\sigma_g} - \frac{1}{|M|} \right), \\ \displaystyle\int_M w_i\, d\sigma_g = 0, \ i = 1, \dots, n. \end{cases} \tag{2.5.25}$$

Problem (2.5.25) is particularly relevant for our purposes, since it describes the limiting problem for a class of $SU(n+1)$-vortices concerning the non-abelian Chern–Simons model of (1.3.99) and (1.3.100), as will be discussed in Chapter 4.

From an analytical point of view, the Toda system (2.5.25) offers the advantage that we can determine explicitly all solutions of the related planar problem:

$$\begin{cases} -\Delta u_i = K_{ij} e^{u_i} \text{ in } \mathbb{R}^2, \\ i = 1, \dots, n \end{cases} \tag{2.5.26}$$

(K_{ij} in (2.5.17)) in terms of n-complex functions, by means of a formula that reduces to Liouville formula (2.2.3) for the case when $n = 1$ (see [Ko], [MOP], and [LS]). By this information, Jost–Wang in [JoW2] were able to obtain a classification result for all solutions of (2.5.26) subject to certain integrability condition, in the same spirit of Theorem 2.2.1 (or Corollary 2.2.2). This has furnished the starting point for the blow-up analysis developed in [JoLW] yielding to some compactness results and degree formulae for solutions of (2.5.25). See [JoLW] for details, and [LN], [ChOS], and [MN] for previous results.

2.6 Final remarks and open problems

We conclude this chapter with a summary of the main open problems concerning the mean field equations of the Liouville-type in (2.5.2), with particular emphasis on those related to the study of Chern–Simons vortices. We start by considering the case for which the weight function $h =$ constant.

Open problem: Let $M = \mathbb{T}^2$ be the flat 2-torus and $\lambda_1(\mathbb{T}^2)$ be the first positive eigenvalue of $-\Delta$ in $H^1(\mathbb{T}^2)$. Is it true that problem (2.5.2) (with $M = \mathbb{T}^2$, $h = 1$) admits *only* the trivial solution, $u = 0$, if and only if

$$0 < \mu < \min\{8\pi, \lambda_1(\mathbb{T}^2)|\mathbb{T}^2|\}? \tag{2.6.1}$$

This property holds for the standard sphere $M = S^2$ (cf. [On]) and has been established for the rectangular torus

$$\mathbb{T}^2 = \mathbb{C} \diagup (a\mathbb{Z} + ib\mathbb{Z}), \quad 0 < a \le b, \tag{2.6.2}$$

only when the ratio $\frac{a}{b}$ is subject to suitable restrictions (see [CLS] and [LiL]) that however allows us to take equality in (2.6.1).

For a more general non-trivial weight function $h \ge 0$, one can equivalently ask: when problem (2.5.2) with $M = \mathbb{T}^2$ and μ satisfying (2.6.1) admits a *unique* solution? Particularly useful to our purposes would be consideration of

$$h = e^{u_0}, \text{ with } u_0 \text{ the solution of (2.5.5).} \tag{2.6.3}$$

Actually, it would be relevant already to resolve such uniqueness issues for minimizers of the functional (2.5.1).

So far, uniqueness of minimizers has been established only for $h =$ constant and with M as in (2.6.2) (cf. [LiL1]).

Another question of interest here concerns the existence of extremals for the "sharp" weighted Moser–Trudinger inequality. This amounts to asking,

Open problem: For which class of non-trivial weight functions $h \ge 0$ does the functional (2.5.1) with $\mu = 8\pi$ attain its infimum in E?

Partial answers to this problem are contained in [NT2], [DJLW1], and [ChL2]; however, these *do not* include the case of interest to us, namely, when h satisfies (2.6.3) with $N = 2$ in (2.5.5) and so $\mu = 4\pi N = 8\pi$. For this double-vortex case, we must specify two vortex points, z_1 and z_2.

The work in [ChLW] and more recently [LiW] and [LiW1] provides a full answer to this question when $z_1 = z_2$, namely when the two vortex points coincide.

In particular, in the case of a torus, $\mathbb{T}^2 = \mathbb{C} \diagup (\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$, we know that: if $\omega_1 = a$ and $\omega_2 = ib$, as in (2.6.2), then the torus is rectangular and no extremal exists; whereas, if we let $\omega_1 = 1$ and $\omega_2 = e^{i\frac{\pi}{3}}$, then an extremal is uniquely attained for the resulting rhombus-shaped torus.

Naturally, we want to know what happens when $z_1 \ne z_2$. More precisely we ask the following:

Open problem: Let $h = e^{u_0}$ with u_0 satisfying (2.5.5), $N = 2$, and $z_1 \ne z_2$. Does $I_{\mu=8\pi}$ attain its infimum in E?

Note that there is a discontinuity of the degree formula at the value $\mu = 8\pi$:

$$d_\mu(\mathbb{T}^2) = \begin{cases} 1, & \text{for } \mu \in (0, 8\pi) \\ 3, & \text{for } \mu \in (8\pi, 16\pi). \end{cases}$$

Similar degree formulae are not available when $\mu \in (16\pi, +\infty) \setminus 8\pi\mathbb{N}$, where more generally, the question of existence of critical points for I_μ in E remains wide-open to investigation.

All questions posed above can also be asked for systems, where much less is known. For instance, the uniqueness issue in the coercive case has not yet been explored, while the existence or non-existence of extremals for the corresponding sharp version of the Moser–Trudinger inequality poses a difficult question answered only in certain cases (see [JoLW]). We shall return to these issues relating to systems at the end of Chapter 4.

3

Planar Selfdual Chern–Simons Vortices

3.1 Preliminaries

In this chapter we investigate the existence of planar Chern–Simons vortices. We start by considering the "pure" Chern–Simons 6th-order model where, on account of (2.1.8) and (2.1.9), we are lead to seek solutions over $\mathbb{R}^2$ for the elliptic equation

$$-\Delta u = \frac{4}{k^2} e^u \left(\nu^2 - e^u\right) - 4\pi \sum_{j=1}^{N} \delta_{z_j} \tag{3.1.1}$$

where $z_1, \ldots, z_N$ are the assigned *vortex points* (not necessarily distinct) corresponding to the zeroes of the Higgs field ϕ.

By means of the transformation,

$$u(z) \to u\left(\frac{z}{\nu^2}\right) - \log \nu^2 \text{ and } z_j \to \nu^2 z_j,\ \ j = 1, \ldots, N, \tag{3.1.2}$$

we can always assume that $\nu^2 = 1$. Hence, after setting

$$\lambda = \frac{4}{k^2}, \tag{3.1.3}$$

we focus our attention on the equation:

$$-\Delta u = \lambda e^u \left(1 - e^u\right) - 4\pi \sum_{j=1}^{N} \delta_{z_j},\ \text{ in } \mathbb{R}^2. \tag{3.1.4}$$

Recalling the relation $e^u = |\phi|^2$ where, in superconductivity theory, $|\phi|$ measures the number density of Cooper pairs, we see that a complete superconductive state is reached when $|\phi| = 1$, while a "mixed state" is attained for $|\phi| < 1$. Therefore, in order to provide physically meaningful "mixed state" vortex-type configurations, we need to supplement (3.1.4) with the condition

$$e^u < 1 \text{ in } \mathbb{R}^2. \tag{3.1.5}$$

By the elliptic regularity theory, it is clear that any weak formulation of (3.1.4) should ensure the following regularity property:

$$u - \sum_{j=1}^{N} \log |z - z_j| \in C^{\infty}\left(\mathbb{R}^2\right). \tag{3.1.6}$$

Therefore, we expect to verify (3.1.5) by means of the maximum principle applied to the smooth function $1 - e^u$, which satisfies (in the classical sense) the equation:

$$\Delta\left(1 - e^u\right) = e^{2u}\left(1 - e^u\right) - e^u|\nabla u|^2 \text{ in } \mathbb{R}^2. \tag{3.1.7}$$

Note that, although $|\nabla u|$ is singular at the vortex points, the term $e^u|\nabla u|^2$ extends smoothly through them.

Furthermore, for the corresponding vortex solution we also need to ensure a finite energy condition, and this amounts to solving (3.1.4) under the constraint:

$$\int_{\mathbb{R}^2} e^u\left(1 - e^u\right) < +\infty. \tag{3.1.8}$$

Condition (3.1.8) identifies two reasonable types of boundary conditions for (3.1.4). First, we may look for solutions such that,

$$e^u \to 1, \text{ as } |z| \to +\infty. \tag{3.1.9}$$

The class of solutions satisfying (3.1.4), (3.1.5), (3.1.8) and (3.1.9) are called *topological* solutions. This terminology is motivated by the fact that in this case the vortex number N takes a topological meaning as it coincides with the degree of ϕ in $\mathbb{R}^2$. In terms of N, we shall express the corresponding value of the "quantized" energy and fluxes. The corresponding topological vortices are asymptotically gauge equivalent (near infinity) to the asymmetric vacua states: $|\phi| = 1$.

Also it makes sense to consider solutions of (3.1.5) subject to the boundary condition

$$e^u \to 0, \text{ as } |z| \to +\infty. \tag{3.1.10}$$

This gives rise to a different class of vortex configurations, called *non-topological* vortices, asymptotically gauge equivalent to the symmetric vacuum state $\phi = 0$. Non-topological vortices present new and interesting features as shall be discussed in Section 3.4. For the moment, let us mention that the distinction between topological and non-topological vortex configuration cannot occur in the abelian (selfdual) Maxwell–Higgs model, where only asymmetric vacua states are present. Hence, Maxwell–Higgs vortices can only be of a topological nature, and have been completely characterized by Taubes in [Ta1] and [JT] in terms of the vortex number N.

Following Wang [Wa] and in the spirit of Taubes' approach, we shall solve (3.1.4), (3.1.5), (3.1.8) and (3.1.9) to obtain selfdual topological vortices for the Chern–Simons 6th-order model, and show that they have much in common with the Maxwell–Higgs vortices described in [Ta1] and [JT].

The existence of non-topological Chern–Simons vortices poses a much harder problem, whose study involves a more detailed analysis of Liouville-type equations.

After some partial results in the radial setting (cf. [SY1]), the existence of planar non-topological solutions (with an assigned set of vortex points) was established by Chae–Imanuvilov in [ChI1]. Subsequently, a different class of physically interesting non-topological solutions were constructed by Chan–Fu–Lin in [CFL]. The solutions in [CFL] exhibit a "bubbling" behavior around the (suitably assigned) vortex points where the magnetic field "concentrates" to carry a flux, which is prescribed and can be arbitrarily large, independent of the vortex number. Such "concentration" behavior occurs also for topological solutions, in such cases however, the "quantized" flux takes a value proportional to the vortex number, and so it cannot be arbitrarily large.

For curiosity, let us mention that if we ignore the finite energy condition (3.1.8), then a solution of (3.1.4) could admit both behaviors (3.1.9) and (3.1.10) along different directions to infinity.

In fact, if we neglect the Dirac measures in (3.1.4) and take $\lambda = 1$, then we check that

$$u(z) = \log\left(\frac{1}{1+|e^z|}\right) < 0 \tag{3.1.11}$$

satisfies the equation,

$$-\Delta u = e^u\left(1-e^u\right) \text{ in } \mathbb{R}^2,$$

(recall (2.2.2)). Furthermore, by means of the polar coordinates $z = \rho e^{i\theta}$, we have

$$e^{u(z)} \to 1 \text{ or } e^{u(z)} \to 0 \text{ as } \rho \to +\infty,$$

according to whether $\theta \in (\frac{\pi}{2}, \frac{3}{2}\pi)$ or $\theta \in (0, \frac{\pi}{2}) \cup (\frac{3}{2}\pi, 2\pi)$. As a matter of fact, from (3.1.11) we can construct a second solution by taking:

$$v(z) = \log\left(1 - e^{u(z)}\right) = \log\frac{|e^z|}{1+|e^z|}.$$

Indeed, we easily check that $\Delta u = \Delta v$ and $e^u(1-e^u) = e^v(1-e^v)$. Note however that the solutions above do not satisfy the finite energy condition (3.1.8), as we can see by a simple change of variables that gives:

$$\int_{\{|\mathrm{Im}\, z|<2\pi\}} e^u\left(1-e^u\right) = \int_{\{|\mathrm{Im}\, z|<2\pi\}} \frac{|e^z|}{(1+|e^z|)^2} = 4\int_{\mathbb{R}^2} \frac{|dw|}{\left(1+|w|^2\right)^2} = 4\pi.$$

Therefore, by the 2π-periodicity of u with respect to the second variable, we derive $e^u(1-e^u) \notin L^1(\mathbb{R}^2)$. This result also applies to the solution v.

This example supports a multiple-existence result for (3.1.4), in addition to the fact that condition (3.1.8) should be used to select only one of the two conditions (3.1.9) and (3.1.10) to hold at infinity.

Before entering into the technical details of the study of (3.1.1), we point out some useful relations between u and the corresponding Chern–Simons vortex. From

Section 2.1 of Chapter 2 we recall that if u is a solution for (3.1.1), then we may use it in (2.1.8), (2.1.9) and (2.1.13) to obtain a vortex configuration $(\mathcal{A}, \phi)_\pm$ solution to (1.2.45) (with the $\pm$ sign choosen accordingly). Since $|\phi_\pm|^2 = e^u$, we shall drop the $\pm$ sign, by simply writing $|\phi|$. Therefore, we see that:

$$\begin{cases} (F_{12})_\pm = \pm\dfrac{2}{k^2}|\phi|^2\left(\nu^2 - |\phi|^2\right) = \dfrac{2}{k^2}e^u\left(\nu^2 - e^u\right) \\ (J_0)_\pm = kF_{12} = \pm\dfrac{2}{k}|\phi|^2\left(\nu^2 - |\phi|^2\right) = \pm\dfrac{2}{k}e^u\left(\nu^2 - e^u\right) \\ (A_0)_\pm = \pm\dfrac{1}{k}\left(\nu^2 - |\phi|^2\right) = \pm\dfrac{1}{k}\left(\nu^2 - e^u\right). \end{cases} \tag{3.1.12}$$

Furthermore, if we use the second (Euler–Lagrange) equation (1.2.34) for our static solution $(\mathcal{A}, \phi)_\pm$, we deduce the relations:

$$(J_1)_\pm = -k\partial_2(A_0)_\pm \text{ and } (J_2)_\pm = k\partial_1(A_0)_\pm;$$

and from (1.2.42), we obtain the following expression for the *selfdual* (static) energy density,

$$\mathcal{E} = \pm\nu^2 F_{12} \mp \frac{1}{2}k\Delta(A_0)_\pm. \tag{3.1.13}$$

In addition, recalling (3.1.7) we see that:

$$\begin{aligned} \pm k\Delta(A_0)_\pm &= \Delta\left(\nu^2 - e^u\right) = \frac{4}{k^2}e^{2u}\left(\nu^2 - e^u\right) - e^u|\nabla u|^2 \\ &= \frac{4}{k^2}|\phi|^4\left(\nu^2 - |\phi|^2\right) - 4|\nabla|\phi||^2. \end{aligned}$$

Consequently from (1.2.38) and by means of (3.1.12) and the selfdual equation (1.2.45) we derive:

$$\begin{aligned} |D_1\phi_\pm|^2 + |D_2\phi_\pm|^2 &= \pm(F_{12})_\pm|\phi|^2 \mp \frac{\varepsilon^{jk}}{2}\partial_j(J_k) = \pm(F_{12})_\pm|\phi|^2 \mp \frac{k}{2}\Delta(A_0)_\pm \\ &= \frac{2}{k^2}|\phi|^4\left(\nu^2 - |\phi|^2\right) - \frac{2}{k^2}|\phi|^4\left(\nu^2 - |\phi|^2\right) + 2|\nabla|\phi||^2 \\ &= 2|\nabla|\phi||^2 = \frac{1}{2}e^u|\nabla u|^2. \end{aligned}$$

3.2 Planar topological Chern–Simons vortices

The goal of this section is to obtain planar *topological* selfdual Chern–Simons vortices.

Namely, we seek solutions $(\mathcal{A}, \phi)$ for (1.2.45) for which

$$\phi : \mathbb{C} \to \mathbb{C},\ \mathcal{A} = -iA_\alpha\, dx^\alpha,\ A_\alpha = A_\alpha\left(x^1, x^2\right) \in \mathbb{R},\ \alpha = 0, 1, 2,$$

and that are consistent from the physical point of view, in the sense that

$$|\phi| < \nu \text{ in } \mathbb{R}^2 \quad \text{and} \quad |\phi|^2 \left(\nu^2 - |\phi|^2\right) \in \mathrm{L}^1\left(\mathbb{R}^2\right), \tag{3.2.1}$$

under the *topological* boundary conditions

$$|\phi(z)| \longrightarrow \nu, \text{ as } |z| \to +\infty. \tag{3.2.2}$$

We prove a result that aims to show that topological Chern–Simons vortices are in direct analogy with the abelian Maxwell–Higgs vortices described by Taubes in [T1], [T2], and [JT], and more generally analyzed in [WY], [Ga1], [Ga2], [Ga3] [Bra1], [Bra2], and the references therein.

For the reader's convenience, we recall Taubes' results on planar Maxwell–Higgs vortices:

Theorem 3.2.1 *([T1]) Given any integer $N \in \mathbb{N}$ and a set $\{z_1, \ldots, z_N\}$ of N points in $\mathbb{R}^2$, there exists a (finite energy) solution $(\mathcal{A}, \phi)_\pm$ of the selfdual equations (1.2.25), (1.2.26), and (1.2.27) (with $\pm$ sign chosen accordingly) that is* unique *up to a gauge equivalence, and has the following properties:*

(i) the solution is globally C^∞
(ii) the zeroes of $\phi_\pm$ coincide with the set $\{z_1, \ldots, z_N\}$ and

$$\phi_+(z) \;\; or \;\; \overline{\phi}_-(z) = O\left((z - z_j)^{n_j}\right) \qquad z \to z_j$$

where $n_j \in \mathbb{N}$ is the multiplicity of z_j in $\{z_1, \ldots, z_N\}$.

Moreover, for suitable $c_0 > 0$,

$$|F_{12}| + |D_1\phi| + |D_2\phi| \le c_0 \left(1 - |\phi|^2\right). \tag{3.2.3}$$

(iii) $\forall\, \epsilon > 0\; \exists\, C_\epsilon > 0$,

$$0 < 1 - |\phi| < C_\epsilon e^{-(1-\epsilon)|z|} \quad in\ \mathbb{R}^2. \tag{3.2.4}$$

(iv)

$$N = \pm\frac{1}{2\pi} \int_{\mathbb{R}^2} (F_{12})_\pm = \frac{1}{\pi}\mathcal{E}.$$

Theorem 3.2.2 *([T2]) Any critical point of the action functional*

$$\frac{1}{2} \int_{\mathbb{R}^2} \left(F_{12}^2 + |D_1\phi|^2 + |D_2\phi|^2\right) + \frac{1}{8} \int_{\mathbb{R}^2} \left(1 - |\phi|^2\right)^2$$

(see (1.2.22)) is given by a solution in Theorem 3.2.1. In particular, mixed vortex-antivortex solutions for the selfdual Maxwell–Higgs equations do not exist.

The above results pertain only to specific values of the relevant parameters, namely, $\lambda = 1$ and $\nu = 1$.

However for the Chern–Simons model, we wish to emphasize the role of such parameters, and in fact obtain estimates (of the type (3.2.4)) which hold uniformly with respect to $\lambda = \frac{4}{k^2}$ (where the Chern–Simons constant satisfies $k > 0$). This fact will enable us to describe the asymptotic behavior of the vortex solution as $k \to 0$.

More precisely we prove:

Theorem 3.2.3 *For any integer* $N \in \mathbb{N}$ *and any assigned set of (vortex) points* $Z = \{z_1, \ldots, z_N\} \subset \mathbb{R}^2$ *(repeated according to their multiplicity) and* $k > 0$*, we find* $(\mathcal{A}, \phi)_\pm$ *a smooth solution to the selfdual equation (1.2.45) in* $\mathbb{R}^2$ *(with the* $\pm$ *sign chosen accordingly) such that the following holds.*

i) Properties of $\phi_\pm$*:*
(a) $\phi_\pm$ *vanishes exactly at the set* Z*. Moreover if* $n_j \in \mathbb{Z}$ *is the multiplicity of* $z_j \in Z$ $j = 1, \ldots, N$*, then*

$$\phi_+(z) \text{ and } \overline{\phi}_-(z) = O\left(\left(z - z_j\right)^{n_j}\right), \text{ as } z \to z_j. \tag{3.2.5}$$

Furthermore, (3.2.1) and (3.2.2) hold for $\phi_\pm$*.*
(b) $|\phi_\pm|$ *is monotone decreasing with respect to* $k > 0$ *and is maximal among all solutions of (1.2.45), satisfying (3.2.2) and (3.2.5).*
(c) For every $k_0 > 0$, $\varepsilon \in (0, 1)$ *and* $\delta > 0$*, there exist constants* $C_\varepsilon > 0$ *and* $C_{\varepsilon,\delta} > 0$ *such that the following estimates hold for every* $0 < k \le k_0$:

$$0 < \nu - |\phi_\pm| \le C_\varepsilon e^{-\frac{2}{k}(1-\varepsilon)|z|} \text{ in } \mathbb{R}^2, \tag{3.2.6}$$

$$\begin{aligned} |D_1\phi_\pm| + |D_2\phi_\pm| + |(F_{12})_\pm| &\le C_{\varepsilon,\delta} e^{-\frac{2}{k}(1-\varepsilon)|z|}, \\ \forall z \in \Omega_\delta &= \mathbb{R}^2 \setminus \cup_{j=1}^N B_\delta(z_j). \end{aligned} \tag{3.2.7}$$

ii) Asymptotic behavior as $k \to 0^+$:
(a) For every $\varepsilon \in (0, 1)$*, we find* $k_\varepsilon > 0$ *such that* $\forall k \in (0, k_\varepsilon)$*, we have*

$$|D_1\phi_\pm| + |D_2\phi_\pm| + |(F_{12})_\pm| \le C_0 e^{\frac{2}{k}(1-\varepsilon)(R_0 - |z|)}, \ \forall |z| \ge R_0,$$

with suitable positive constants c_0*,* C_0 *and* R_0 *independent of* ε *and* k*.*
(*b*) *For* $m \in \mathbb{Z}^+$ *and* $\delta > 0$*, we have*

$$\|\, |D_1\phi_\pm| + |D_2\phi_\pm| + |(F_{12})_\pm| \,\|_{C^m(\Omega_\delta)} \le \frac{c_0}{k^2} \|\, \nu - |\phi| \,\|_{C^m(\Omega_\delta)} \to 0,$$

as $k \to 0^+$ *with suitable* $c_0 > 0$ *independent of* k*. Moreover the above convergence to zero holds faster than any power of* k*.*
(c) As $k \to 0^+$,

$$(A_0)_\pm \to 0, \ \left(J^0\right)_\pm \to 0 \text{ in } \mathrm{L}^1\left(\mathbb{R}^2\right);$$

and

$$(F_{12})_\pm \to \pm 2\pi \sum_{j=1}^N \delta_{z_j}, \tag{3.2.8}$$

$$\left((A_0)_\pm\right)^2 \to \pi \sum_{j \in J} n_j^2 \delta_{z_j} \text{ and } \frac{1}{k}(A_0)_\pm \to \pm\pi \sum_{j \in J} n_j(n_j + 1)\delta_{z_j},$$

weakly in the sense of measure in $\mathbb{R}^2$*. Here* $J \subset \{1, \ldots, N\}$ *is a set of indices identifying all of the different vortices in Z. In particular,*

$$||(A_0)_\pm||^2_{L^2(\mathbb{R}^2)} \to \pi \sum_{j \in J} n_j^2 \text{ and } ||\frac{1}{k}(A_0)_\pm||_{L^1(\mathbb{R}^2)} \to \pi \sum_{j \in J} n_j \left(n_j + 1\right).$$

iii) (Quantization) The following holds respectively for the magnetic flux, the electric charge and the total energy:

$$\begin{aligned} &\text{Magnetic flux } \Phi = \int_{\mathbb{R}^2} (F_{12})_\pm = \pm 2\pi N; \\ &\text{Electric charge } Q = \int_{\mathbb{R}^2} \left(J^0\right)_\pm = \pm 2\pi k N; \\ &\text{Total energy } E = \int_{\mathbb{R}^2} \mathcal{E}_\pm = 2\pi v^2 N. \end{aligned} \tag{3.2.9}$$

Theorem 3.2.3 above summarizes and improves several of the results available in literature concerning topological Chern–Simons vortices (see e.g., [Wa], [Y1], and [Ha3]).

It furnishes the analogous version of Theorem 3.2.1 in the context of Chern–Simons theory. In fact, we observe that the Maxwell–Higgs vortices (of Theorem 3.2.1) also satisfy the "concentration" property (3.2.8) as established in [HJS].

It is believed that, in analogy with Maxwell–Higgs vortices, Chern–Simons topological vortex configurations are *uniquely* determined once the location of their vortex points has been specified. In other words, the *maximal* Chern–Simons vortex configuration described above is the *only* solution of (1.2.45) satisfying (3.2.2) and (3.2.5).

This fact has been established for the case in which all vortex points coincide, namely $z_1 = z_2 = \cdots = z_N$ (see [Ha3]). Indeed, in this case the magnitude $|\phi|^2$ of the Higgs field can be shown to be radially symmetric about the (multiple) vortex point (cf. [Ha3]) and uniqueness follows from a general result about the radial solutions established by Chen–Hastings–McLeod–Yang in [CHMcLY].

In Section 3.3 we shall prove that uniqueness holds for small values of the parameter $k > 0$ (depending on the assigned set Z of vortex points), provided that all solutions of (1.2.45), (3.2.2), and (3.2.5) also satisfy uniform (exponential) decay estimates, as those claimed in (3.2.6) and (3.2.7) for the "maximal" solution. Recently, such uniform bounds have been established by K. Choe [Cho], who deduces a uniqueness result when $k > 0$ is either very small or very large (in dependence of the set Z). See also [ChN] for related results.

Still, the question of uniqueness remains *open* for all values of $k > 0$ and any assigned set Z of vortex points.

Also we mention that it is not known whether a pointwise estimate of the type (3.2.3) is valid for solutions of the Chern–Simons selfdual equations (1.2.45) in $\mathbb{R}^2$.

To obtain Theorem 3.2.3, we shall construct a solution u for (3.1.4) with the appropriate asymptotic properties at infinity.

To this purpose, we shall simultaneously account for the (singular) behavior of u at the vortex points and at infinity, by taking u of the form

$$u(z) = u_0(z) + \upsilon(z) \tag{3.2.10}$$

with

$$u_0(z) = \sum_{j=1}^{N} \log\left(\frac{|z - z_j|^2}{1 + |z - z_j|^2}\right). \tag{3.2.11}$$

Note that, in virtue of (2.2.2), u_0 satisfies:

$$-\Delta u_0 = \sum_{j=1}^{N} \frac{4}{\left(1 + |z - z_j|^2\right)^2} - 4\pi \sum_{j=1}^{N} \delta_{z_j}. \tag{3.2.12}$$

Therefore, for the (smooth) new unkown υ the problem is reduced to solve:

$$-\Delta \upsilon = \lambda e^{u_0+\upsilon}\left(1 - e^{u_0+\upsilon}\right) - \sum_{j=1}^{N} \frac{4}{\left(1 + |z - z_j|^2\right)^2} \text{ in } \mathbb{R}^2 \tag{3.2.13}$$

$$\upsilon(z) \to 0, \quad \text{as } |z| \to +\infty. \tag{3.2.14}$$

We set,

$$g_0(z) = 4\sum_{j=1}^{N} \frac{1}{\left(1 + |z - z_j|^2\right)^2}, \tag{3.2.15}$$

and, note that $g_0 \in C^\infty(\mathbb{R}^2) \cap \mathrm{L}^p(\mathbb{R}^2)$, $1 \le p \le +\infty$. So any (weak) solution υ of (3.2.13), (3.2.14) satisfies $\upsilon \in C^\infty(\mathbb{R}^2) \cap \mathrm{L}^\infty(\mathbb{R}^2)$.

We observe the following:

Proposition 3.2.4 *Let υ be a (smooth) solution for (3.2.13). Then υ satisfies (3.2.14) if and only if*

$$\upsilon \in H^1\left(\mathbb{R}^2\right). \tag{3.2.16}$$

Furthermore for $u = u_0 + \upsilon$ (the solution of (3.1.4)) the following holds:
i)

$$u \in \mathrm{L}^p\left(\mathbb{R}^2\right)\ \forall p \ge 1, \ \textit{and}\ u < 0 \ \textit{in}\ \mathbb{R}^2; \tag{3.2.17}$$

ii)

$$e^u\left(1 - e^u\right) \in \mathrm{L}^1\left(\mathbb{R}^2\right) \ \textit{and}\ \lambda \int_{\mathbb{R}^2} e^u\left(1 - e^u\right) = 4\pi N. \tag{3.2.18}$$

Proof. Assume that v satisfies (3.2.14). We start to check that $u = u_0+v \le 0$ in $\mathbb{R}^2$ and find that the more strict inequality claimed in (3.2.17) follows from the Hopf lemma. To this end, we argue by contradiction and suppose that $\sup_{\mathbb{R}^2} u > 0$. Since $u \to 0$ as $|z| \to +\infty$, and $u(z) \to -\infty$ as $z \to z_j$ for every $j = 1, \ldots N$, we see that u attains its (positive) maximum value at a point $z_* \in \mathbb{R}^2 \setminus Z$ such that, $u_* := u(z_*) > 0$ and $\Delta u(z_*) \le 0$. But this is impossible, since by (3.1.4) we are lead to the contradiction:

$$0 \le -\Delta u(z_*) = \lambda e^{u_*} \left(1 - e^{u_*}\right) < 0;$$

and so

$$u < 0 \text{ in } \mathbb{R}^2. \tag{3.2.19}$$

Next we verify that (3.2.18) holds. To this purpose, let χ denote the standard cut-off function defined by the properties:

$$\chi \in C_0^\infty(\mathbb{R}^2),\ \chi = 1 \text{ in } B_1(0),\ \chi = 0 \text{ in } \mathbb{R}^2 \setminus B_2(0) \text{ and } 0 \le \chi \le 1. \tag{3.2.20}$$

Then for $R > 0$, set $\chi_R(z) := \chi(\frac{z}{R})$ and use this function as a test function in (3.2.13) to find

$$-\int_{\mathbb{R}^2} v\, \Delta \chi_R = \lambda \int_{\mathbb{R}^2} e^{u_0+v} \left(1 - e^{u_0+v}\right) \chi_R - \int_{\mathbb{R}^2} g_0 \chi_R. \tag{3.2.21}$$

Observe that, as $R \to +\infty$,

$$\int_{\mathbb{R}^2} g_0 \chi_R \to \int_{\mathbb{R}^2} g_0 = 4 \sum_{j=1}^N \int_{\mathbb{R}^2} \frac{1}{\left(1 + |z - z_j|^2\right)^2} = 4\pi N, \tag{3.2.22}$$

and by means of (3.2.14),

$$\left| \int_{\mathbb{R}^2} v\, \Delta \chi_R \right| = \left| \int_{\{R \le |z| \le 2R\}} v\, \Delta \chi_R \right| \le ||v||_{\mathrm{L}^\infty(R \le |z| \le 2R)} \int_{\{1 \le |z| \le 2\}} |\Delta \chi| \to 0.$$

Therefore from (3.2.19) and (3.2.21) and the dominated convergence theorem we deduce that, $e^{u_0+v}(1 - e^{u_0+v}) \in \mathrm{L}^1(\mathbb{R}^2)$, and $\lambda \int_{\mathbb{R}^2} e^{u_0+v}(1 - e^{u_0+v}) = 4\pi N$. We see next how to use (3.2.18) to show that $v \in H^1(\mathbb{R}^2)$. For $\delta > 0$ sufficiently small, set

$$\Omega_\delta = \mathbb{R}^2 \setminus \cup_{j=1}^N B_\delta(z_j), \tag{3.2.23}$$

so that $u \in \mathrm{L}^\infty(\Omega_\delta)$. For $z \in \Omega_\delta$, we can use the elementary inequality:

$$|1 - e^t| \ge \frac{|t|}{1 + |t|},\ \forall t \in \mathbb{R} \tag{3.2.24}$$

to estimate

$$|u(z)| \le \left(1 + ||u||_{\mathrm{L}^\infty(\Omega_\delta)}\right) |1 - e^{u(z)}| \le \left(1 + ||u||_{\mathrm{L}^\infty(\Omega_\delta)}\right) e^{||u||_{\mathrm{L}^\infty(\Omega_\delta)}} e^{u(z)} \left(1 - e^{u(z)}\right).$$

Thus, from (3.2.18), we may actually conclude that $u \in \mathrm{L}^1(\Omega_\delta) \cap \mathrm{L}^\infty(\Omega_\delta)$. On the other hand, since $|u| \leq |u_0| + |v| \in \mathrm{L}^p(B_\delta(z_j))$, $\forall p \geq 1, \forall j = 1, \ldots, N$ we see that

$$u \in \mathrm{L}^p\left(\mathbb{R}^2\right), \ \forall p \geq 1. \tag{3.2.25}$$

Notice that for u_0 in (3.2.11) we have

$$u_0 \in \mathrm{L}^1_{\mathrm{loc}}\left(\mathbb{R}^2\right) \cap \mathrm{L}^q\left(\mathbb{R}^2\right), \ \forall q > 1. \tag{3.2.26}$$

Therefore, by combining (3.2.25) and (3.2.26) we obtain,

$$v \in \mathrm{L}^q\left(\mathbb{R}^2\right), \ \forall 1 < q \leq \infty. \tag{3.2.27}$$

This information allow us to use $v\chi_R$ as a test function in (3.2.13), and for

$$f = \lambda e^u\left(1 - e^u\right) - g_0, \tag{3.2.28}$$

we find:

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla v|^2 \chi_R &= -\int_{\mathbb{R}^2} v \nabla v \nabla \chi_R + \int_{\mathbb{R}^2} f v \chi_R = -\frac{1}{2}\int_{\mathbb{R}^2} \nabla v^2 \nabla \chi_R + \int_{\mathbb{R}^2} f v \chi_R \\ &= \frac{1}{2}\int_{\mathbb{R}^2} v^2 \Delta \chi_R + \int_{\mathbb{R}^2} f v \chi_R.\end{aligned}$$

As $R \to +\infty$,

$$\left|\int_{\mathbb{R}^2} v^2 \Delta \chi_R\right| \leq ||v||^2_{\mathrm{L}^\infty(R \leq |z| \leq 2R)} \int_{\{1 \leq |z| \leq 2\}} |\Delta \chi| \to 0$$

and

$$\int_{\mathbb{R}^2} f v \chi_R \to \int_{\mathbb{R}^2} f v.$$

Consequently, we may conclude that $|\nabla v| \in \mathrm{L}^2(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} |\nabla v|^2 = \int_{\mathbb{R}^2} f v.$$

In particular, $v \in H^1(\mathbb{R}^2)$ as claimed.

Vice versa, assume that $v \in H^1(\mathbb{R}^2)$. Since we may express f in (3.2.28) as

$$\begin{aligned} f &= \lambda\left(1 - e^{u_0 + v}\right) - \lambda\left(1 - e^{u_0 + v}\right)^2 - g_0 \\ &= \lambda\left(2e^{u_0}\left(1 - e^{u_0}\right)\left(e^v - 1\right) - \left(1 - e^{u_0}\right)^2 - e^{2u_0}\left(1 - e^v\right)^2\right) - g_0, \end{aligned}$$

with $e^{u_0} \in L^\infty(\mathbb{R}^2)$ and $(1 - e^{u_0}) \in L^p(\mathbb{R}^2)$, $\forall 1 < p \leq +\infty$, then by Lemma 2.4.7 we see that $f \in L^p(\mathbb{R}^2)$ $\forall p \geq 2$. Thus by well-known elliptic estimates (e.g., see Corollary 9.21 in [GT]), we find a constant $C > 0$ such that,

$$\sup_{B_1(z)} |v| \leq C \left(||v||_{L^2(B_2(z))} + ||f||_{L^2(B_2(z))} \right), \ \forall z \in \mathbb{R}^2.$$

Consequently, for $R > 1$,

$$\sup_{|z| \geq R+1} |v| \leq C \left(||v||_{L^2(|z| \geq R-1)} + ||f||_{L^2(|z| \geq R-1)} \right) \to 0, \ \text{ as } R \to +\infty$$

and (3.2.14) is established. □

Remark 3.2.5 We wish to observe that the mere assumption

$$u = u_0 + v \in L^p(\{|z| \geq r_0\}), \ \text{for some } p \geq 1 \text{ and } r_0 > 0$$

suffices to ensure that a solution v of (3.2.13) also satisfies (3.2.14). This follows easily from the observation that $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ define subharmonic functions in $\mathbb{R}^2 \setminus Z$. Thus, by the mean value theorem we deduce that:

$$u(z) \to 0, \ \text{as } |z| \to +\infty.$$

In the same way, the condition

$$u = u_0 + v \in L^\infty(\{|z| \geq r_0\}), \ \text{for some } r_0 > 0, \tag{3.2.29}$$

implies $u(z) \to 0$, as $|z| \to +\infty$, whenever v satisfies (3.2.13) and

$$e^{u_0+v} \left(1 - e^{u_0+v}\right) \in L^1\left(\mathbb{R}^2\right). \tag{3.2.30}$$

Indeed, we can use, as above, the inequality (3.2.24) together with (3.2.28) and (3.2.30), to see that $u_0 + v \in L^1(\{|z| \geq r_0\})$, from which (3.2.14) follows.

Notice that if we assume (3.2.30), then we may still deduce (3.2.14) under the weaker condition: $(u_0 + v)^- \in L^p(\{|z| \geq r_0\})$ for some $1 \leq p \leq +\infty$ and $r_0 > 0$. Indeed (3.2.30) allow us to control the positive part $(u_0+v)^+$ by means of the estimate:

$$0 \leq (u_0 + v)^+ \leq e^{(u_0+v)^+} \left|1 - e^{(u_0+v)^+}\right| \leq e^{u_0+v} \left|1 - e^{u_0+v}\right| \in L^1\left(\mathbb{R}^2\right).$$

Next, at fixed $\lambda > 0$, we provide some useful exponential decay estimates at infinity for solutions of (3.2.13), (3.2.14).

Proposition 3.2.6 *Let v be a (classical) solution for (3.2.13), (3.2.14), and set $u = u_0 + v$. For every $\varepsilon \in (0, 1)$ and $\delta > 0$, there exist suitable constants $C_\varepsilon = C_\varepsilon(\lambda) > 0$ and $C_{\varepsilon,\delta} = C_{\varepsilon,\delta}(\lambda) > 0$, such that*

$$\begin{aligned} &(i)\ 0 < 1 - e^{u(z)} \leq C_\varepsilon e^{-(1-\varepsilon)\sqrt{\lambda}|z|} \text{ in } \mathbb{R}^2; \\ &(ii)\ |u(z)| + |\nabla u(z)| \leq C_{\varepsilon,\delta} e^{-(1-\varepsilon)\sqrt{\lambda}|z|}, \ \forall z \in \Omega_\delta = \mathbb{R}^2 \setminus \cup_{j=1}^N B_\delta(z_j). \end{aligned} \tag{3.2.31}$$

Proof. Recall that f in (3.2.28) satisfies

$$||f||_{L^1(\mathbb{R}^2)} \le 8\pi N, \ ||f||_{L^\infty(\mathbb{R}^2)} \le \lambda + 4N, \tag{3.2.32}$$

and since v is bounded, we can use Green's representation formula to write

$$\nabla v(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{z-y}{|z-y|^2} f(y).$$

Consequently, for every $R > 0$ we estimate:

$$\begin{aligned} |\nabla v(z)| &\le \frac{1}{2\pi}\left(\int_{\{|z-y|>R\}} \frac{1}{|y-z|}|f(y)| + \int_{\{|z-y|<R\}} \frac{1}{|y-z|}|f(y)|\right) \\ &\le \frac{1}{R}\frac{1}{2\pi}||f||_{L^1(\mathbb{R}^2)} + R||f||_{L^\infty(\mathbb{R}^2)}. \end{aligned}$$

So, if we optimize the estimate above with respect to R, we find:

$$||\nabla v||_{L^\infty(\mathbb{R}^2)} \le 2\sqrt{\frac{1}{2\pi}||f||_{L^1(\mathbb{R}^2)}||f||_{L^\infty(\mathbb{R}^2)}}. \tag{3.2.33}$$

Moreover, for $r_0 := 2\max\{|z_j|, \ j = 1, \dots, N\} + 1$, we have

$$|\nabla u_0(z)| \le 2\sum_{j=1}^{N} \frac{1}{|z-z_j|\left(1+|z-z_j|^2\right)} \le \frac{8N}{|z|^3}, \text{ for } |z| > r_0,$$

and with the help of (3.2.32) and (3.2.33), we derive

$$|\nabla(u_0+v)(z)| \le 8N\left(\frac{1}{|z|^3} + \sqrt{1+\frac{\lambda}{4N}}\right), \ \forall |z| > r_0. \tag{3.2.34}$$

To obtain (3.2.31), we introduce the function

$$\psi(z) = e^{A(R-|z|)} + e^{u_0+v} - 1,$$

with $R > r_0$ and A to be specified below. Clearly, $\psi \ge 0$ in B_R. To analyze what happens for $|z| \ge R$, we observe that ψ satisfies the following boundary value problem:

$$\begin{cases} \Delta\psi = \left(A^2 - \frac{A}{|z|} - \lambda e^{2(u_0+v)}\right) e^{A(R-|z|)} + \lambda e^{2(u_0+v)}\psi + e^{u_0+v}|\nabla(u_0+v)|^2 \\ \psi|_{\partial B_R} \text{ and } \psi(z) \to 0, \text{ as } |z| \to +\infty. \end{cases} \tag{3.2.35}$$

Therefore, if we suppose that $\inf_{|z|\ge R} \psi < 0$, then we find $z_R \in \mathbb{R}^2$ such that $|z_R| > R$ and $\psi(z_R) = \inf_{|z|\ge R} \psi < 0$. In particular, $\nabla\psi(z_R) = 0$ leads to the identity

$$e^{(u_0+v)(z_R)}|\nabla(u_0+v)(z_R)|^2 = \langle -Ae^{A(R-|z_R|)}\frac{z_R}{|z_R|}, \nabla(u_0+v)(z_R)\rangle, \quad (3.2.36)$$

and $\Delta\psi(z_R) \geq 0$. Using (3.2.34) into (3.2.36), we derive the estimate

$$e^{(u_0+v)(z_R)}|\nabla(u_0+v)(z_R)|^2 \leq Ae^{A(R-|z_R|)}8N\left(\frac{1}{|z|^3}+\sqrt{1+\frac{\lambda}{4N}}\right),$$

which we can insert into (3.2.35) to find

$$\begin{aligned} 0 &\leq \Delta\psi(z_R) \\ &\leq \left(A^2 - \tfrac{A}{|z_R|} - \lambda e^{2(u_0+v)(z_R)} + 8NA\left(\tfrac{1}{|z_R|^3}+\sqrt{1+\tfrac{\lambda}{4N}}\right)\right)e^{A(R-|z_R|)}. \end{aligned} \quad (3.2.37)$$

Notice that, $A_0 := \sqrt{16N^2+(4N+1)\lambda} - 4N\sqrt{1+\frac{\lambda}{4N}}$ satisfies

$$A_0^2 + 8NA_0\sqrt{1+\frac{\lambda}{4N}} = \lambda;$$

and for every $A \in (0, A_0)$ there exists a unique $\delta \in (0, 1)$,

$$A^2 + 8NA\sqrt{1+\frac{\lambda}{4N}} = \lambda(1-\delta).$$

Recalling that $e^{u_0+v} \to 1$ as $|z| \to +\infty$, we reach a contradiction in (3.2.37) with a sufficiently large $R > 0$ satisfying,

$$R^2 > 8N \text{ and } \inf_{|z|\geq R} e^{2(u_0+v)} > 1-\delta.$$

In other words, we have established that,

$$\begin{aligned} &\forall A \in (0, A_0) \text{ there exists a constant } C_A > 0 \text{ such that,} \\ &1 - e^{(u_0+v)(z)} \leq C_A e^{-A|z|}, \text{ in } \mathbb{R}^2. \end{aligned} \quad (3.2.38)$$

Since $A_0 < \sqrt{\frac{\lambda}{4N}}$, the estimate (3.2.38) comes up short in covering the claimed estimate (3.2.31). We show next, that (3.2.38) actually holds with $A = (1-\varepsilon)\sqrt{\lambda}$, $\forall\varepsilon \in (0, 1)$.

To this purpose, observe that from (3.2.38) we have

$$|(u_0+v)(z)| + \lambda e^{(u_0+v)(z)}\left(1 - e^{(u_0+v)(z)}\right) \leq C_\delta e^{-A|z|}, \ \forall z \in \Omega_\delta \quad (3.2.39)$$

for every $A \in (0, A_0)$ and a suitable constant $C_\delta > 0$, which now depends also on $\delta > 0$.

Since $-\Delta(u_0 + v) = \lambda e^{u_0+v}(1 - e^{u_0+v})$ in Ω_δ, we can use (3.2.39) together with the well-known gradient estimates for Poisson's equation (e.g., see Theorem 3.9 in [GT]), to obtain:

$$|\nabla(u_0 + v)(z)| \le Ce^{-A|z|}, \ \forall z \in \Omega_{2\delta} = \mathbb{R}^2 \setminus \cup_{j=1}^N B_{2\delta}(z_j). \tag{3.2.40}$$

But if we substitute (3.2.40) into (3.2.36), we can improve (3.2.37); and for $R > 0$ sufficiently large, we can see that

$$\inf_{|z|\ge R} \psi < 0 \text{ implies: } A^2 - \lambda \inf_{|z|\ge R} e^{2(u_0+v)} \ge 0.$$

Since for any $A \in (0, \sqrt{\lambda})$ we can easily contradict the latter of the inequalities above by letting $R \to +\infty$, we arrive at the desired estimate of *(i)* in (3.2.31).

In turn, for $\forall \varepsilon \in (0, 1)$ we can take $A = (1 - \varepsilon)\sqrt{\lambda}$ in (3.2.39), and consequently in (3.2.40), and thus obtain estimate *(ii)* in (3.2.31). □

We now turn to construct a solution for (3.2.13), (3.2.14).

On the basis of Proposition 3.2.4, we know that $H^1(\mathbb{R}^2)$ furnishes the appropriate functional space in which to search for the solution.

In fact, in $H^1(\mathbb{R}^2)$ we can formulate our problem in a variational form by considering the functional,

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^2} \left(e^{u_0+v} - 1\right)^2 + \int_{\mathbb{R}^2} g_0 v, \ v \in H^1\left(\mathbb{R}^2\right).$$

By virtue of Lemma 2.4.7, we know that $I_\lambda \in C^1(H^1(\mathbb{R}^2))$. Moreover, on the basis of the elliptic regularity theory, any critical point of I_λ in $H^1(\mathbb{R}^2)$ furnishes a classical solution for (3.2.13), which by Proposition 3.2.4, also satisfies (3.2.14).

Thus, we only need to focus on the search for critical points for I_λ in $H^1(\mathbb{R}^2)$. In this direction we obtain:

Proposition 3.2.7 *For* $\lambda > 8||g_0||^2_{L^2(\mathbb{R}^2)}$ *the functional* I_λ *is bounded from below and coercieve in* $H^1(\mathbb{R}^2)$. *It attains an infimum at* $v_\lambda \in H^1(\mathbb{R}^2)$ *that satisfies:*

$$||u_0 + v_\lambda||_{L^2(\mathbb{R}^2)} \to 0, \ \text{as } \lambda \to +\infty. \tag{3.2.41}$$

Proof. We can use the elementary inequality (3.2.24) to estimate

$$\int_{\mathbb{R}^2} \left(1 - e^{u_0+v}\right)^2 \ge \int_{\mathbb{R}^2} \frac{(u_0 + v)^2}{(1 + |(u_0 + v)|)^2} \ge \frac{1}{2}\int_{\mathbb{R}^2} \frac{v^2}{(1 + |v|)^2} - g\int_{\mathbb{R}^2} |u_0|^2,$$

and find for any $v \in H^1(\mathbb{R}^2)$,

$$I_\lambda(v) \ge \frac{1}{2}||\nabla v||^2_{L^2(\mathbb{R}^2)} + \frac{\lambda}{4}\int_{\mathbb{R}^2} \frac{v^2}{\left(1 + |v|^2\right)^2} - \frac{g\lambda}{2}||u_0||^2_{L^2(\mathbb{R}^2)} - ||g_0||^2_{L^2(\mathbb{R}^2)}||v||^2_{L^2(\mathbb{R}^2)}. \tag{3.2.42}$$

On the other hand, by means of Sobolev's well-known interpolation inequality,

$$\int_{\mathbb{R}^2} v^4 \leq 2\left(\int_{\mathbb{R}^2} v^2\right)\left(\int_{\mathbb{R}^2} |\nabla v|^2\right), v \in H^1\left(\mathbb{R}^2\right),$$

we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^2} v^2\right)^2 &\leq \left(\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2}\right)\left(\int_{\mathbb{R}^2} v^2(1+|v|)^2\right)\\ &\leq 2\left(\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2}\right)\left(\int_{\mathbb{R}^2} v^2\right)\left(1+2||\nabla v||^2_{L^2(\mathbb{R}^2)}\right)\end{aligned}$$

and arrive at the inequality

$$||v||_{L^2(\mathbb{R}^2)} \leq \left(2\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2}\right)^{\frac{1}{2}}\left(1+2||\nabla v||^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}}. \tag{3.2.43}$$

In view of (3.2.42), this implies:

$$\begin{aligned}I_\lambda(v) \geq{}& \frac{1}{2}||\nabla v||^2_{L^2(\mathbb{R}^2)} + \frac{\lambda}{4}\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2}\\ &-||g_0||_{L^2(\mathbb{R}^2)}\left(2\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2}\left(1+2||\nabla v||^2_{L^2(\mathbb{R}^2)}\right)\right)^{\frac{1}{2}} - g\lambda||u_0||^2_{L^2(\mathbb{R}^2)}\\ \geq{}& \frac{1}{2}||\nabla v||^2_{L^2(\mathbb{R}^2)} + \frac{\lambda}{4}\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2} - \frac{\varepsilon}{2}\left(1+2||\nabla v||^2_{L^2(\mathbb{R}^2)}\right)\\ &-\frac{1}{\varepsilon}||g_0||^2_{L^2(\mathbb{R}^2)}\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2} - g\lambda||u_0||^2_{L^2(\mathbb{R}^2)}\\ ={}& \left(\frac{1}{2}-\varepsilon\right)||\nabla v||^2_{L^2(\mathbb{R}^2)} + \left(\frac{\lambda}{4} - \frac{||g_0||^2_{L^2(\mathbb{R}^2)}}{\varepsilon}\right)\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2} - \frac{\varepsilon}{2}\\ &-g\lambda||u_0||^2_{L^2(\mathbb{R}^2)},\end{aligned}$$

for $v \in H^1(\mathbb{R}^2)$ and for sufficiently small $\varepsilon > 0$. Therefore, for $\lambda > 8||g_0||^2_{L^2(\mathbb{R}^2)}$ we may fix $\varepsilon_\lambda > 0$ such that $\delta_\lambda = \frac{1}{2} - \varepsilon_\lambda > 0$ and $\sigma_\lambda = \frac{\lambda}{4} - \frac{||g_0||^2_{L^2(\mathbb{R}^2)}}{\varepsilon_\lambda} > 0$. We obtain

$$I_\lambda(v) \geq \delta_\lambda||\nabla v||^2_{L^2(\mathbb{R}^2)} + \sigma_\lambda\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2} - C_\lambda,\ \forall v \in H^1\left(\mathbb{R}^2\right), \tag{3.2.44}$$

for suitable $C_\lambda > 0$.

In this way, we have established that I_λ is bounded from below in $H^1(\mathbb{R}^2)$, provided $\lambda > 8||g_0||^2_{L^2(\mathbb{R}^2)}$.

Denote by $||v|| = \left(||\nabla v||^2_{L^2} + ||v||^2_{L^2}\right)^{\frac{1}{2}}$ the norm in $H^1(\mathbb{R}^2)$ using (3.2.43) into (3.2.44), we find

$$
\begin{aligned}
I_\lambda(v) &\geq \delta_\lambda ||\nabla v||^2_{L^2(\mathbb{R}^2)} + \frac{\sigma_\lambda}{2} \frac{||v||^2_{L^2(\mathbb{R}^2)}}{1 + 2||\nabla v||^2_{L^2(\mathbb{R}^2)}} - C_\lambda \\
&= \delta_\lambda ||\nabla v||^2_{L^2(\mathbb{R}^2)} + \frac{\sigma_\lambda}{2} \frac{||v||^2_{L^2(\mathbb{R}^2)} + ||\nabla v||^2_{L^2(\mathbb{R}^2)}}{1 + 2||\nabla v||^2_{L^2(\mathbb{R}^2)}} - C_\lambda \\
&\geq \sqrt{\frac{1}{2}\delta_\lambda \sigma_\lambda (1 + 2||v||^2)} - \beta_\lambda
\end{aligned}
$$

for suitable $\beta_\lambda > 0$; and this proves the coerciveness of I_λ.

Since I_λ is weakly lower semicontinuous in $H^1(\mathbb{R}^2)$, we conclude that, for $\lambda > 8||g_0||^2_{L^2(\mathbb{R}^2)}$, the functional I_λ attains an infimum at a point $v_\lambda \in H^1(\mathbb{R}^2)$.

It remains to establish (3.2.41).

Claim: There exist suitable constants $a_0, b_0 > 0$ such that $\forall \lambda > 8||g_0||^2_{L^2(\mathbb{R}^2)}$ we have

$$
I_\lambda(v_\lambda) = \inf_{H^1(\mathbb{R}^2)} I_\lambda \leq a_0 \log \lambda + b_0. \tag{3.2.45}
$$

To obtain (3.2.45), we shall evaluate I_λ over a suitable test function. To simplify notation, we take the vortex points $z_1, \ldots, z_N$ to be distinct, since with the obvious modifications we can treat also the case of multiple-vortex points.

Let $\varepsilon > 0$ but sufficiently small (to be specified below), such that the balls $B_\varepsilon(z_j)$ are mutually distinct. Set

$$
v_\varepsilon(z) = \begin{cases} -u_0(z) & z \in \mathbb{R}^2 \setminus \cup_{j=1}^N B_\varepsilon(z_j) \\ -\left(\log \frac{\varepsilon^2}{1+\varepsilon^2} + \sum_{k \neq j} \log \frac{|z-z_k|^2}{1+|z-z_k|^2}\right) & z \in B_\varepsilon(z_j),\ j = 1, \ldots, N. \end{cases} \tag{3.2.46}
$$

So that $v_\varepsilon \in H^1(\mathbb{R}^2)$ and,

$$
u_0(z) + v_\varepsilon(z) = \begin{cases} 0 & z \in \mathbb{R}^2 \setminus \cup_{j=1}^N B_\varepsilon(z_j) \\ \log\left(\left|\frac{z-z_j}{\varepsilon}\right|^2 \frac{1+\varepsilon^2}{1+|z-z_j|^2}\right) & z \in B_\varepsilon(z_j),\ j = 1, \ldots, N. \end{cases} \tag{3.2.47}
$$

We estimate,

$$
\begin{aligned}
I_\lambda(v_\varepsilon) &= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla v_\varepsilon|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^2}(1-e^{u_0+v_\varepsilon})^2 + \int_{\mathbb{R}^2} g_0 v_\varepsilon \\
&\le \sum_{j=1}^{N}\Bigg[c_N\int_{\{|z-z_j|\ge\varepsilon\}}\frac{1}{|z-z_j|^2(1+|z-z_j|^2)^2} \\
&\quad +\frac{\lambda}{2}\int_{\{|z-z_j|<\varepsilon\}}\left(1-\left|\frac{z-z_j}{\varepsilon}\right|^2\frac{1+\varepsilon^2}{1+|z-z_j|^2}\right)^2 \\
&\quad +\int_{\{|z-z_j|<\varepsilon\}} g_0(z)\log\left(\left|\frac{z-z_j}{\varepsilon}\right|^2\frac{1+\varepsilon^2}{1+|z-z_j|^2}\right)\Bigg] + C_1 \qquad (3.2.48) \\
&\le C_2\left(\int_{\{|z|>\varepsilon\}}\frac{1}{|z|^2(1+|z|^2)^2} + \lambda\int_{\{|z|<\varepsilon\}}\left(1-\left|\frac{z}{\varepsilon}\right|^2\frac{1+\varepsilon^2}{1+|z|^2}\right)^2\right. \\
&\quad \left. +\int_{\{|z|<\varepsilon\}}\left|\log\left|\frac{z}{\varepsilon}\right|^2\right| + 1\right) \\
&= C_3\left(\log\frac{1}{\varepsilon} + \lambda\varepsilon^2\int_0^1 r\left(1-r^2\frac{1+\varepsilon^2}{1+\varepsilon^2 r}\right)^2 dr + \varepsilon^2\int_0^1 r\log\frac{1}{r^2}\,dr + 1\right),
\end{aligned}
$$

with suitable positive constants c_N, C_1, C_2 and C_3 that are independent of λ and ε. Consequently, by choosing $\varepsilon^2 = \frac{1}{\lambda}$, we conclude (3.2.45).

At this point, we can use (3.2.45) to deduce the estimates

$$
||\nabla v_\lambda||^2_{\mathrm{L}^2(\mathbb{R}^2)} \le c_0(\log\lambda + ||u_0+v_\lambda||_{\mathrm{L}^2(\mathbb{R}^2)} + 1), \qquad (3.2.49)
$$

$$
\begin{aligned}
\int_{\mathbb{R}^2}\frac{(u_0+v_\lambda)^2}{(1+|u_0+v_\lambda|)^2} &\le \int_{\mathbb{R}^2}(1-e^{u_0+v_\lambda})^2 \\
&\le \frac{c_0}{\lambda}\left(\log\lambda + ||u_0+v_\lambda||_{\mathrm{L}^2(\mathbb{R}^2)} + 1\right) \qquad (3.2.50)
\end{aligned}
$$

for a suitable constant $c_0 > 0$ and independent of λ. Consequently,

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^2}(u_0+v_\lambda)^2\right)^2 \\
&\le \left(\int_{\mathbb{R}^2}\frac{(u_0+v_\lambda)^2}{(1+|u_0+v_\lambda|)^2}\right)\left(\int_{\mathbb{R}^2}(u_0+v_\lambda)^2(1+|u_0+v_\lambda|)^2\right) \qquad (3.2.51) \\
&\le \left(\int_{\mathbb{R}^2}\frac{(u_0+v_\lambda)^2}{(1+|u_0+v_\lambda|)^2}\right)\left(||u_0+v_\lambda||^2_{\mathrm{L}^2(\mathbb{R}^2)} + 8||u_0||^4_{\mathrm{L}^4(\mathbb{R}^2)} + 8\int_{\mathbb{R}^2} v_\lambda^4\right),
\end{aligned}
$$

and

$$\begin{aligned}\int_{\mathbb{R}^2} v_\lambda^4 &\le 2\|v_\lambda\|^2_{L^2(\mathbb{R}^2)}\|\nabla v_\lambda\|^2_{L^2(\mathbb{R}^2)} \\ &\le 4\|u_0+v_\lambda\|^2_{L^2(\mathbb{R}^2)}\|\nabla v_\lambda\|^2_{L^2(\mathbb{R}^2)} + \|u_0\|^2_{L^2(\mathbb{R}^2)}\|\nabla v_\lambda\|^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{3.2.52}$$

So, we can insert (3.2.52) into (3.2.51), to find a suitable constant $C > 0$ (and independent of λ) such that,

$$\begin{aligned}&\|u_0+v_\lambda\|^4_{L^2(\mathbb{R}^2)} \\ &\le C\left(\int_{\mathbb{R}^2}\frac{|u_0+v_\lambda|^2}{1+|u_0+v_\lambda|^2}\right)\left(\|u_0+v_\lambda\|^2_{L^2(\mathbb{R}^2)}+1\right)\left(\|\nabla v_\lambda\|^2_{L^2(\mathbb{R}^2)}+1\right).\end{aligned} \tag{3.2.53}$$

Thus, substituting (3.2.49) and (3.2.50) into (3.2.53) we see that,

$$\|u_0+v_\lambda\|_{L^2(\mathbb{R}^2)} \to 0, \text{ as } \lambda \to +\infty. \qquad \square$$

Corollary 3.2.8 *For every $\lambda > 0$, problem (3.1.4), (3.1.9) admits a maximal solution $u_\lambda < 0$, monotone increasing with respect to the parameter $\lambda > 0$, and satisfying:*

$$\|u_\lambda\|_{L^2(\mathbb{R}^2)} \to 0, \text{ as } \lambda \to +\infty. \tag{3.2.54}$$

Proof. Recall that by *maximal* we mean that u_λ satisfies

$$u_\lambda(z) \ge u(z), \ \forall z \in \mathbb{R}^2 \setminus Z,$$

for any solution u of (3.1.4), (3.1.9).

Let us first treat the case where $\lambda > 8\|g_0\|^2_{L^2(\mathbb{R}^2)}$ (as required in Proposition 3.2.7), and set,

$$v_-(z) = \sup\{v(z) : v \in H^1(\mathbb{R}^2) \text{ solves (3.2.13), (3.2.14)}\} \le -u_0(z).$$

So, $v_- \in H^1(\mathbb{R}^2)$ defines a (bounded) *subsolution* for (3.2.13), (3.2.14), which yields to the existence of a solution $\hat{v}_\lambda \in H^1(\mathbb{R}^2)$ for (3.2.13), (3.2.14) satifying: $\hat{v}_\lambda \ge v_-$ in $\mathbb{R}^2$ (see e.g., Theorem 2.4 in [St1]). Consequently, $u_\lambda = u_0 + \hat{v}_\lambda$ defines a maximal solution for (3.1.4), (3.1.9) and in particular,

$$0 > u_\lambda(z) \ge (u_0+v_\lambda)(z), \text{ a.e. in } \mathbb{R}^2$$

with v_λ given in Proposition 3.2.7. Therefore,

$$\|u_\lambda\|_{L^2(\mathbb{R}^2)} \le \|u_0+v_\lambda\|_{L^2(\mathbb{R}^2)} \to 0, \text{ as } \lambda \to +\infty.$$

For $0 < \lambda_1 < \lambda_2$, the function $\hat{v}_{\lambda_1}$ defines a strict subsolution for (3.2.13). Consequently, for (3.2.14) with $\lambda = \lambda_2$, we deduce the necessary condition: $\hat{v}_{\lambda_1}(z) < \hat{v}_{\lambda_2}(z)\ \forall z \in \mathbb{R}^2$; and the desired conclusion follows in this case.

Now to eliminate the condition $\lambda > 8\|g_0\|^2_{L^2(\mathbb{R}^2)}$, we observe that for a given $\varepsilon > 0$, nothing changes in our analysis if we replace u_0 in (3.2.11) with $u_{0,\varepsilon}(z) = \sum_{j=1}^N \log\left(\frac{|z-z_j|^2}{\frac{1}{\varepsilon}+|z-z_j|^2}\right)$; thus $g_0(z)$ in (3.2.15) becomes $g_{0,\varepsilon}(z) = 4\sum_{j=1}^N \left(\frac{1}{\varepsilon}+|z-z_j|^2\right)^{-2}$. Therefore the above conclusion holds, provided that $\lambda > 8\|g_{0,\varepsilon}\|^2_{L^2(\mathbb{R}^2)} = O(\varepsilon^2)$. We can always verify this condition by choosing $\varepsilon > 0$ sufficiently small. □

We mention that the first existence result for (3.1.4), (3.1.9) was established in [Wa], while we refer to [Y1] for an alternative proof based on an iteration scheme.

Property (3.2.54) has the interesting consequence of implying a strong "localization" property for the solution u_λ around the vortex points (as $\lambda \to +\infty$), as is consistent with physical applications.

More precisely the following holds:

Proposition 3.2.9 *Let $p_0 \geq 1$ and $\lambda_0 \geq 0$. Assume that we have a solution v_λ of (3.2.13) such that $u_\lambda = u_0 + v_\lambda$ (solution of (3.1.4)) satisfies:*

$$u_\lambda \in L^{p_0}(\mathbb{R}^2) \text{ and } \|u_\lambda\|_{L^{p_0}(\mathbb{R}^2)} \leq C, \ \forall \lambda \geq \lambda_0$$

with a suitable $C > 0$ (independent of λ). Then
(i) for every $\varepsilon \in (0,1)$ and $\delta > 0$ the estimates of (3.2.31) hold with uniform constants C_ε and $C_{\varepsilon,\delta}$ independent of $\lambda \geq \lambda_0$.
(ii) As $\lambda \to +\infty$:

$$\|u_\lambda\|_{L^p(\mathbb{R}^2)} \to 0, \ \forall p \geq 1; \tag{3.2.55}$$

$$\lambda e^{u_\lambda}(1 - e^{u_\lambda}) \to 4\pi \sum_{j=1}^N \delta_{z_j}, \text{ weakly in the sense of the measure in } \mathbb{R}^2. \tag{3.2.56}$$

Furthermore, for any $\varepsilon \in (0,1)$ there exists a constant $\lambda_\varepsilon \geq \lambda_0$, such that $\forall \lambda \geq \lambda_\varepsilon$ the following uniform estimates hold:
(iii)

$$|\nabla u_\lambda(z)| \leq \lambda c_0 \left(1 - e^{u_\lambda(z)}\right) \leq C_0 e^{(1-\varepsilon)\sqrt{\lambda}(R_0 - |z|)}, \ \forall |z| \geq R_0; \tag{3.2.57}$$

with suitable constants $R_0 > 0$, $c_0 > 0$, and $C_0 > 0$ that are independent of λ and ε.
(iv)

$$\lambda^s \|u_\lambda\|_{C^m(\Omega_\delta)} \to 0, \text{ as } \lambda \to +\infty, \tag{3.2.58}$$

for every $s, m \in \mathbb{Z}^+$ and $\delta > 0$, where $\Omega_\delta = \mathbb{R}^2 \setminus \cup_{j=1}^N B_\delta(z_j)$.

Proof. First of all, observe that by Proposition 3.2.4 and Remark 3.2.5 we know that

$$u_\lambda \to 0 \text{ as } |z| \to +\infty, \ e^{u_\lambda} < 1 \text{ in } \mathbb{R}^2 \text{ and } \lambda \int_{\mathbb{R}^2} e^{u_\lambda}(1 - e^{u_\lambda}) = 4\pi N. \tag{3.2.59}$$

Hence, u_λ defines a superharmonic function in $\mathbb{R}^2 \backslash \{z_1, \ldots, z_N\}$. Thus, for every $\delta > 0$ and $z \in \Omega_\delta = \mathbb{R}^2 \setminus \cup_{j=1}^N B_\delta(z_j)$, we can use the mean value theorem to get

$$|u_\lambda(z)| = -u_\lambda(z) \leq \frac{1}{\pi\delta^2} \int_{B_\delta(z)} |u_\lambda|. \tag{3.2.60}$$

Consequently,

$$||u_\lambda(z)||_{L^\infty(\Omega_{2\delta})} \leq \left(\frac{1}{\pi\delta^2}\right)^{\frac{1}{p_0}} ||u_\lambda||_{L^{p_0}(\Omega_\delta)} \leq \frac{C}{\pi\delta^2}. \tag{3.2.61}$$

By means of (3.2.24), (3.2.59), and (3.2.61), for any $p \geq 1$ we find,

$$\begin{aligned} \int_{\Omega_\delta} |u_\lambda|^p &\leq (1 + ||u_\lambda||_{L^\infty(\Omega_\delta)})^p \int_{\Omega_\delta} (1 - e^{u_\lambda})^p \\ &\leq (1 + ||u_\lambda||_{L^\infty(\Omega_\delta)})^p e^{||u_\lambda||_{L^\infty(\Omega_\delta)}} \int_{\Omega_\delta} e^{u_\lambda}(1 - e^{u_\lambda}) \leq \frac{c_{\delta,p}}{\lambda} \end{aligned} \tag{3.2.62}$$

with a suitable constant $c_{\delta,p} > 0$ depending on δ and p but independent of λ. In particular, using (3.2.62) with $p = 1$ and (3.2.60), we can improve (3.2.61) and conclude

$$||u_\lambda||_{L^1(\Omega_\delta)} + ||u_\lambda||_{L^\infty(\Omega_\delta)} \leq \frac{C_\delta}{\lambda}, \tag{3.2.63}$$

and consequently obtain

$$\lambda||e^{u_\lambda}(1 - e^{u_\lambda})||_{L^\infty(\Omega_\delta)} \leq C_\delta \tag{3.2.64}$$

with a suitable constant $C_\delta > 0$ that is independent of λ.

To analyze what happens around the vortex points, we observe that v_λ admits $L^{p_0}_{\mathrm{loc}}(\mathbb{R}^2)$-norm uniformly bounded in λ. Also since Δv_λ is uniformly bounded in $L^1(\mathbb{R}^2)$ and $\Delta v_\lambda \leq g_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, we can use well-known elliptic estimates to ensure that v_λ is locally and uniformly bounded from below. In particular, for $\forall\delta > 0$ we find a constant $d_\delta > 0$ and

$$|u_\lambda| = -u_\lambda \leq |u_0| + d_\delta, \text{ in } D_\delta := \cup_{j=1}^N B_\delta(z_j).$$

On the other hand, by (3.2.63) we know that $u_\lambda \to 0$ pointwise a.e., as $\lambda \to +\infty$. So, by dominated convergence, for any $p \geq 1$: $||u_\lambda||_{L^p(D_\delta)} \to 0$ as $\lambda \to +\infty$, and this together with (3.2.63) yields to (3.2.55).

Furthermore, observe that from (3.2.63) and (3.2.64) we also know that both v_λ and Δv_λ are bounded in $L^\infty(\Omega_\delta)$ and uniformly in λ. Therefore, the well-known elliptic and Sobolev's estimates imply that $|\nabla v_\lambda|$ is also uniformly bounded in $L^\infty(\Omega_\delta)$, and for suitable positive constants c_1 and r_1, the following holds:

$$|u_\lambda(z)| \leq \frac{c_1}{\lambda R^2} \text{ and } |\nabla u_\lambda(z)| \leq \frac{8N}{|z|^3} + \frac{c_1}{R^2}, \ \forall |z| > R, \tag{3.2.65}$$

for every $R \geq r_1$. Let $R_0 := \max\{r_1, \sqrt{8N}, |z_j|\ j = 1, \ldots, N\}+1$, and for $\varepsilon \in (0, 1)$, $R \geq R_0$ consider the function

$$\psi_\lambda(z) = \frac{C_1}{\lambda} e^{(1-\varepsilon)\sqrt{\lambda}(R-|z|)} + e^{u_0+v_\lambda} - 1$$

with C_1 choosen (depending only on R) in such a way that,

$$\psi_\lambda \geq 0 \text{ in } \partial B_R.$$

Notice that we can always attain the property above by virtue of (3.2.63).

Since $\psi_\lambda(z) \to 0$ as $|z| \to +\infty$, we can apply to ψ_λ the arguments given in the proof of Proposition 3.2.6, and using (3.2.65) conclude that,

$$\text{if } \inf_{|z|\geq R} \psi_\lambda < 0, \text{ then necessarily } (1-\varepsilon)^2 + \frac{c_1}{R^2\sqrt{\lambda}}(1-\varepsilon) - \inf_{|z|\geq R} e^{2u_\lambda} \geq 0.$$

On the basis of (3.2.63) and (3.2.65), the inequality above is certainly violated either for a fixed $\lambda \geq \lambda_0$ and for $R = R_\varepsilon > 0$ choosen sufficiently large (according to ε), or for $R = R_0$ and for λ sufficiently large (according to ε). In the first case, we deduce the uniform estimates as claimed in (i). In the second case, we find $\lambda_\varepsilon > 0$ such that $\forall \lambda \geq \lambda_\varepsilon$,

$$0 < 1 - e^{u_\lambda(z)} \leq \frac{C_1}{\lambda} e^{(1-\varepsilon)\sqrt{\lambda}(R_0-|z|)}, \ \forall |z| \geq R_0. \tag{3.2.66}$$

Consequently, $\forall z :\ |z| \geq R_0$ we have

$$\begin{aligned} |u_\lambda(z)| + |\Delta u_\lambda(z)| &= |u_\lambda(z)| + \lambda e^{u_\lambda}(1 - e^{u_\lambda}) \leq (1+\lambda)(1 - e^{u_\lambda}) \\ &\leq C_0 e^{(1-\varepsilon)\sqrt{\lambda}(R_0-|z|)} \end{aligned}$$

with suitable $C_0 > 0$ that is independent of λ and ε. Thus, using the well-known gradient estimates for Poisson's equation we conclude (3.2.57).

From (3.2.66) we also have

$$\begin{aligned} \lambda \int_{\{|z|\geq 2R_0\}} e^{u_\lambda}(1 - e^{u_\lambda}) &\leq C_0 \int_{\{|z|\geq 2R_0\}} e^{-(1-\varepsilon)\sqrt{\lambda}\frac{|z|}{2}} \\ &\leq \frac{C_\varepsilon}{\sqrt{\lambda}} e^{-(1-\varepsilon)\sqrt{\lambda}R_0}, \ \lambda \geq \lambda_\varepsilon \end{aligned} \tag{3.2.67}$$

for any given $\varepsilon \in (0, 1)$ and a corresponding suitable constant $C_\varepsilon > 0$ that depends on ε only.

In particular,

$$\lambda \int_{\{|z|\geq 2R_0\}} e^{u_\lambda}(1 - e^{u_\lambda}) \to 0 \text{ as } \lambda \to +\infty. \tag{3.2.68}$$

On the other hand, $\forall \varphi \in C_0^\infty(\mathbb{R}^2)$, we find that

$$\lambda \int_{\mathbb{R}^2} e^{u_\lambda}(1 - e^{u_\lambda})\varphi - 4\pi \sum_{j=1}^{N} \varphi(z_j) = -\int_{\mathbb{R}^2} \Delta u_\lambda \varphi = -\int_{\mathbb{R}^2} u_\lambda \Delta\varphi \to 0, \tag{3.2.69}$$

as $\lambda \to +\infty$, and that (3.2.56) follows from (3.2.68) and (3.2.69). In addition, if we take in (3.2.69) a function $\varphi \in C_0^\infty(B_{3R_0})$ such that $\varphi = 0$ in $\cup_{j=1}^N B_{\frac{\delta}{2}}(z_j)$ while $\varphi = 1$ in $\Omega_\delta \cap B_{2R_0}$, then we obtain

$$\lambda \int_{\Omega_\delta \cap B_{2R_0}} e^{u_\lambda}(1 - e^{u_\lambda}) \leq c_\delta ||u_\lambda||_{\mathrm{L}^1(\Omega_{\frac{\delta}{2}})} \tag{3.2.70}$$

with $c_\delta > 0$ a suitable constant depending on $\delta > 0$ only.

We can use (3.2.67) and (3.2.70) to improve inequalities (3.2.63) and (3.2.64). Indeed, arguing as in (3.2.62) we find

$$\begin{aligned} ||u_\lambda||_{\mathrm{L}^1(\Omega_\delta)} &\leq C_{1,\delta}\left(\int_{\Omega_\delta \cap B_{2R_0}} e^{u_\lambda}(1 - e^{u_\lambda}) + \int_{\{|z| \geq 2R_0\}} e^{u_\lambda}(1 - e^{u_\lambda})\right) \\ &\leq \frac{C_{2,\delta}}{\lambda} ||u_\lambda||_{\mathrm{L}^1(\Omega_{\frac{\delta}{2}})} + \frac{C_{\varepsilon,\delta}}{\sqrt{\lambda}} e^{-(1-\varepsilon)\sqrt{\lambda}R_0} \leq \frac{1}{\lambda^2} C_\delta, \end{aligned}$$

where we have used (3.2.63) to derive the last inequality with a suitable constant $C_\delta > 0$.

Iterating the argument above, we see how to estimate the $\mathrm{L}^1(\Omega_\delta)$-norm of u_λ (and hence its $\mathrm{L}^\infty(\Omega_\delta)$-norm), for any power of $\frac{1}{\lambda}$. In other words, for λ sufficiently large, we have

$$||u_\lambda||_{\mathrm{L}^1(\Omega_\delta)} + ||u_\lambda||_{\mathrm{L}^\infty(\Omega_\delta)} + ||\lambda e^{u_\lambda}(1 - e^{u_\lambda})||_{\mathrm{L}^\infty(\Omega_\delta)} \leq \frac{C}{\lambda^s}, \tag{3.2.71}$$

for any $s \geq 1$, and a suitable positive constant $C = C(\delta, s)$ independent of λ. Recall that $-\Delta u_\lambda = \lambda e^{u_\lambda}(1 - e^{u_\lambda})$ in Ω_δ. So by well-known gradient estimates for Poisson's equation (cf. [GT]) from (3.2.71), follows an analogous L^∞-estimate for $|\nabla u_\lambda|$ in Ω_δ.

At this point, by familiar bootstrap arguments, we arrive at the estimate

$$||u_\lambda||_{C^m(\Omega_\delta)} \leq \frac{C}{\lambda^s}, \ \lambda > 0 \text{ large}; \tag{3.2.72}$$

for any $s \geq 1$ and a suitable constant $C = C(\delta, s, m) > 0$ independent of λ. Clearly, (3.2.58) follows from (3.2.72), as $\lambda \to +\infty$. □

The results of Proposition 3.2.7 and 3.2.9 can be summarized as follows:

Corollary 3.2.10 *For $\lambda > 0$, problem (3.1.4) admits a* **maximal** *solution u_λ monotonically increasing in $\lambda > 0$, such that,*
i) Topological behavior:

(a) $u_\lambda < 0$ *in* $\mathbb{R}^2$, $u_\lambda(z) \to 0$ *as* $|z| \to +\infty$, *and* $\lambda \int_{\mathbb{R}^2} e^{u_\lambda}(1 - e^{u_\lambda}) = 4\pi N$.

(b) For every $\lambda > 0$, $\varepsilon \in (0, 1)$, and $\delta > 0$, there exist contants $C_\varepsilon > 0$ and $C_{\varepsilon,\delta} > 0$ (independent of λ) such that for $\lambda \geq \lambda_0$ we have

$$0 < (1 - e^{u_\lambda}) \le C_\varepsilon e^{-(1-\varepsilon)\sqrt{\lambda}|z|}, \ \forall z \in \mathbb{R}^2, \tag{3.2.73}$$

and

$$|u_\lambda(z)| + |\nabla u_\lambda(z)| \le C_{\varepsilon,\delta} e^{-(1-\varepsilon)\sqrt{\lambda}|z|}, \ \forall z \in \Omega_\delta. \tag{3.2.74}$$

ii) Asymptotic behavior as $\lambda \to +\infty$:

u_λ satisfies properties (3.2.55)–(3.2.58) *of Proposition 3.2.9.*

Furthermore,

$$\lambda(1 - e^{u_\lambda})^2 \to 4\pi \sum_{j\in J} n_j^2 \delta_{z_j} \text{ and } \lambda(1 - e^{u_\lambda}) \to 4\pi \sum_{j\in J} n_j(n_j + 1)\delta_{z_j}, \tag{3.2.75}$$

weakly in the sense of measure in $\mathbb{R}^2$.

In particular,

$$\lambda \int_{\mathbb{R}^2} (1 - e^{u_\lambda})^2 \to 4\pi \sum_{j\in J} n_j^2 \text{ and } \lambda \int_{\mathbb{R}^2} (1 - e^{u_\lambda}) \to 4\pi \sum_{j\in J} n_j(n_j + 1),$$

where $J \subset \{1, \dots, N\}$ is a set of indices identifying all distinct points in $\{z_1, \dots, z_N\}$ and n_j is the multiplicity of z_j, for $j \in J$.

Proof. It remains only to show (3.2.75), which we shall establish by proving that, $\forall \delta > 0$ small,

$$\lambda \int_{B_\delta(z_j)} (1 - e^{u_\lambda})^2 \to 4\pi n_j^2, \text{ as } \lambda \to +\infty.$$

Then, (3.2.75) follows by means of properties (3.2.56)–(3.2.58) and the identity:

$$1 - e^{u_\lambda} = (1 - e^{u_\lambda})^2 + e^{u_\lambda}(1 - e^{u_\lambda}).$$

For simplicity and without loss of generality, we take $z_j = 0$. Note that, for $\delta > 0$ small, we have that $u_\lambda(z) = 2n_j \log|z| +$ (smooth function in $B_\delta(0)$); and therefore,

$$z \cdot \nabla u_\lambda(z) = 2n_j + \varphi_\lambda(z),$$

with φ_λ smooth (around the origin) and $\varphi_\lambda(0) = 0$. Hence, we can obtain a Pohozaev-type identity (see (5.2.14)) by multipling the equation by $z \cdot \nabla u_\lambda(z)$ and integrating over $B_\delta(0)$, as follows:

$$\begin{aligned} \int_{|z|<\delta} -\Delta u_\lambda \nabla u_\lambda(z) \cdot z &= \lambda \int_{|z|<\delta} e^{u_\lambda}(1 - e^{u_\lambda}) \nabla u_\lambda \cdot z - 8\pi n_j^2 \\ &= -\frac{\lambda}{2} \int_{|z|<\delta} \operatorname{div}\left(z(1 - e^{u_\lambda})^2\right) + \lambda \int_{|z|<\delta} (1 - e^{u_\lambda})^2 - 8\pi n_j^2 \\ &= -\frac{\lambda}{2}\delta \int_{|z|=\delta} (1 - e^{u_\lambda})^2 \, d\sigma + \lambda \int_{|z|<\delta} (1 - e^{u_\lambda})^2 - 8\pi n_j^2. \end{aligned} \tag{3.2.76}$$

Using the uniform estimates in (3.2.58) we see that,

$$\lambda \int_{|z|=\delta} (1 - e^{u_\lambda})^2 \, d\sigma = o(1), \text{ as } \lambda \to +\infty. \tag{3.2.77}$$

Next, for what concerns the left-hand side of (3.2.76), denote by $n = \frac{z}{|z|}$ the outward normal of the ball $\{z : |z| \le \delta\}$. Taking into account the singularity of ∇u_λ at z_j, we have

$$\begin{aligned}
&-\int_{|z|\le\delta} \Delta u_\lambda \nabla u_\lambda(z) \cdot z \\
&= -\int_{|z|\le\delta} \operatorname{div}\left(\nabla u_\lambda \nabla u_\lambda(z) \cdot z\right) + \frac{1}{2}\int_{|z|\le\delta} \operatorname{div}\left(z|\nabla u_\lambda|^2\right) \\
&= -\int_{|z|=\delta} (\nabla u_\lambda \cdot z)(\nabla u_\lambda \cdot n)\, d\sigma + \frac{1}{2}\int_{|z|=\delta} |\nabla u_\lambda|^2 n \cdot z \, d\sigma - 4\pi n_j^2 \\
&= \delta \int_{|z|=\delta} \left(\frac{1}{2}|\nabla u_\lambda|^2 - \left(\frac{\partial u_\lambda}{\partial n}\right)^2\right) d\sigma - 4\pi n_j^2 \to -4\pi n_j^2, \text{ as } \lambda \to +\infty,
\end{aligned} \tag{3.2.78}$$

as follows from (3.2.58). Hence passing to the limit as $\lambda \to +\infty$ in (3.2.76), from (3.2.77) and (3.2.78), we conclude that

$$\lambda \int_{|z|\le\delta} (1 - e^{u_\lambda})^2 = 4\pi n_j^2 + o(1), \text{ as } \lambda \to +\infty;$$

and (3.2.75) follows. □

Remark 3.2.11 Notice that, more generally, property (3.2.75) remains valid for any family of solutions u_λ satisfying the assumptions of Proposition 3.2.9.

Proof of Theorem 3.2.3. Let us first establish the desired statement when the symmetry breaking parameter $\nu = 1$. For $k > 0$, set $\lambda = \frac{4}{k^2}$; then it suffices to substitute the *maximal* solution u_λ of Corollary 3.2.10 into (2.1.8), (2.1.9) and (2.1.13) to obtain $\phi_\pm$ and $(A_\alpha)_\pm$, $\alpha = 0, 1, 2$ which defines a solution for the Chern–Simons selfdual equations (1.2.45), with $|\phi_\pm|^2 = e^{u_\lambda}$. Therefore, using (3.1.12) (with $u = u_\lambda$) we check the validity of properties (i), (ii) and (iii) directly by means of the properties of u_λ as claimed in Corollary 3.2.10 (recall that $\nu = 1$ in this case). In particular, from (3.2.59) we see that,

$$\int_{\mathbb{R}^2} (F_{12})_\pm = \pm 2\pi N \text{ and } \int_{\mathbb{R}^2} \left(J^0\right)_\pm = k \int_{\mathbb{R}^2} (F_{12})_\pm = \pm 2\pi k N.$$

To evaluate the total energy, we observe that for $R > 0$,

$$\begin{aligned}
\left| \pm k \int_{\{|z| \leq R\}} \Delta(A_0)_\pm \right| &= \left| \pm k \int_{\{|z|=R\}} \frac{\partial}{\partial n} (A_0)_\pm \, d\sigma \right| \\
&= \left| \int_{\{|z|=R\}} \nabla(1 - e^{u_\lambda}) \cdot \frac{z}{|z|} \, d\sigma \right| \\
&= \left| \int_{\{|z|=R\}} e^{u_\lambda} \nabla u_\lambda \frac{z}{|z|} \, d\sigma \right| \\
&\leq 2\pi R \max_{|z|=R} |\nabla u_\lambda| \to 0, \text{ as } R \to +\infty;
\end{aligned}$$

as it follows from the gradient estimates of u_λ in (3.2.57). Therefore, $\Delta(A_0)_\pm \in \mathrm{L}^1(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} \Delta(A_0)_\pm = 0$. Hence, from (3.1.13) we conclude that,

$$\int_{\mathbb{R}^2} \mathcal{E} = \pm \int_{\mathbb{R}^2} F_{12} = 2\pi N,$$

and this completes the proof of Theorem 3.2.3 when $\nu = 1$. To treat general values of the parameter ν, we simply take the solution u_λ of Corollary 3.2.10 with vortex points located at $\nu^2 z_j$, $j = 1, \ldots, N$. The desired vortex configuration is constructed as above by using the scaled function

$$u_\lambda(\nu^2 z) + 2 \log \nu,$$

which satisfies (3.1.1) together with all the desired properties. □

3.3 A uniqueness result

We conclude the analysis of (3.1.4), (3.1.9) by discussing uniqueness of topological solutions.

We start to mention that uniqueness holds when all vortex points coincide, say with the origin. If we denote by $N \in \mathbb{N}$ the vortex multiplicity, then this situation is described by the problem

$$\begin{cases} -\Delta u = e^u (1 - e^u) - 4\pi N \delta_{z=0} \\ u(z) \to 0 \text{ as } |z| \to +\infty, \end{cases} \tag{3.3.1}$$

as we can scale out the parameter λ by the change of variables $z \to \sqrt{\lambda} z$.

On the basis of a result of Han [Ha3] (see also [CFL]), we know that (3.3.1) admits a unique solution u radially symmetric about the origin.

Moreover, according to the results established in the previous section, we also know that u satisfies

$$u \in \mathrm{L}^p(\mathbb{R}^2) \ \forall p \geq 1, \ u < 0 \text{ in } \mathbb{R}^2 \text{ and } \int_{\mathbb{R}^2} e^u (1 - e^u) = 4\pi N, \tag{3.3.2}$$

and $\forall \varepsilon \in (0, 1)$, there exists a constant $C_\varepsilon > 0$ such that

$$|u(z)| + |\nabla u(z)| + \left(1 - e^{u(z)}\right) \le C_\varepsilon e^{-(1-\varepsilon)|z|}, \ \forall |z| \ge 1. \tag{3.3.3}$$

Furthermore, we can use Pohozaev's identity (as in (3.2.76)–(3.2.78)) together with the estimates above to find:

$$\int_{\mathbb{R}^2} (1 - e^u)^2 = 4\pi N^2 \text{ and } \int_{\mathbb{R}^2} (1 - e^u) = 4\pi N(1 + N). \tag{3.3.4}$$

We show that u is non-degenerate in the following sense:

Theorem 3.3.12 *Let u be the unique radially symmetric solution for (3.3.1). Then the (linearized) problem*

$$\begin{cases} -\Delta\varphi + e^u(2e^u - 1)\varphi = 0 \\ \varphi \in H^1(\mathbb{R}^2), \end{cases} \tag{3.3.5}$$

admits only the trivial solution $\varphi = 0$.

Proof. Arguing as in Proposition 3.2.7, for $\varepsilon_0 > 0$ sufficiently small we can decompose $u = u_0 + v$ in such a way that $u_0(r) = N \log \frac{r^2}{\varepsilon_0^2 + r^2}$ and $v \in H^1(\mathbb{R}^2)$ correspond to the (global) minimum for the functional,

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^2} \left(e^{u_0+v} - 1\right)^2 + \int_{\mathbb{R}^2} g_0 v,$$

with $g_0(z) = g_0(|z|) = \dfrac{4\pi\varepsilon_0^2}{(\varepsilon_0^2 + |z|^2)^2}$. Consequently,

$$\int_{\mathbb{R}^2} |\nabla\varphi|^2 + \int_{\mathbb{R}^2} e^u(2e^u - 1)\varphi^2 = \left(I''(v)\varphi, \varphi\right) \ge 0, \ \varphi \in H^1\left(\mathbb{R}^2\right). \tag{3.3.6}$$

So, any possible non-trivial solution for (3.3.5) would correspond to the *first* eigenfunction relative to the (first) eigenvalue (equal to zero) for the linear operator: $L = -\Delta + e^u(2e^u - 1)$ in $H^1(\mathbb{R}^2)$. Since u is radially symmetric, then necessarily also the first eigenfuction φ must be radially symmetric. Therefore, if we argue by contradiction and let $\varphi \neq 0$ be a solution for (3.3.5) then $\varphi = \varphi(r)$ and without loss of generality, we can also assume $\varphi > 0$. Thus, summarizing the properties above, we have that $u = u(r)$ satisfies

$$\begin{cases} \ddot{u}(r) + \frac{1}{r}\dot{u}(r) = e^{u(r)}(e^{u(r)} - 1), & \forall r > 0 \\ u(r) = 2N \log r + O(1), & \text{as } r \to 0^+ \\ 0 < 1 - e^{u(r)} \le C_\varepsilon e^{-(1-\varepsilon)r}, & \forall r > 0 \end{cases} \tag{3.3.7}$$

with $\varepsilon \in (0, 1)$ and $C_\varepsilon > 0$ a suitable constant depending on ε only (see (3.2.31)).

While for $\varphi = \varphi(r)$ we have:

$$\begin{cases} \ddot{\varphi} + \frac{1}{r}\dot{\varphi} = e^{u(r)}(2e^{u(r)} - 1)\varphi(r) \\ \varphi > 0 \\ \int_0^{+\infty} (\dot{\varphi}^2 + \varphi^2) r\, dr < +\infty. \end{cases} \tag{3.3.8}$$

We make a convenient change of variables and set,

$$U(t) = u(e^t) \text{ and } \psi(t) = \varphi(e^t). \tag{3.3.9}$$

Properties (3.3.7) read as

$$\ddot{U} = e^{2t} e^{U(t)} \left(e^{U(t)} - 1\right) \text{ in } \mathbb{R} \tag{3.3.10}$$

$$U(t) = 2Nt + O(1) \text{ as } t \to -\infty, \tag{3.3.11}$$

$$U(t) < 0 \text{ in } \mathbb{R} \text{ and } \left(1 - e^{U(t)}\right) \leq C_\varepsilon e^{-(1-\varepsilon)e^t}, \text{ as } t \to +\infty \tag{3.3.12}$$

for $\varepsilon \in (0, 1)$ and $C_\varepsilon > 0$ a suitable constant depending on ε only. Hence $\ddot{U} < 0$ in $\mathbb{R}^2$ and so $\dot{U}$ is monotone decreasing in $\mathbb{R}$ and as $t \to \pm\infty$ we find

$$\lim_{t \to -\infty} \dot{U}(t) = \lim_{t \to -\infty} \frac{U(t)}{t} = 2N, \tag{3.3.13}$$

as follows from (3.3.11). Moreover,

$$\lim_{t \to +\infty} \dot{U}(t) = \lim_{t \to +\infty} \frac{U(t)}{t} = 0, \tag{3.3.14}$$

as follows from (3.3.12). While from (3.3.8) we have

$$\begin{cases} \ddot{\psi}(t) = e^{2t} e^{U(t)} \left(2e^{U(t)} - 1\right) \psi \text{ in } \mathbb{R} \\ \psi > 0, \text{ in } \mathbb{R} \text{ and } \int_{-\infty}^{+\infty} \left(\dot{\psi}^2 + \psi^2 e^{2t}\right) dt < +\infty. \end{cases} \tag{3.3.15}$$

Hence, $\ddot{\psi} < 0$ for $t \to -\infty$ while $\ddot{\psi} > 0$ for $t \to +\infty$. Therefore $\dot{\psi}$ is a monotone function for $|t|$ large, so it admits a limit for $t \to \pm\infty$, which must vanish by the integrability property in (3.3.15). Namely,

$$\lim_{t \to \pm\infty} \dot{\psi}(t) = 0 = \lim_{t \to \pm\infty} \frac{\psi(t)}{t}. \tag{3.3.16}$$

As a consequence we find that $\dot{\psi} < 0$ as $t \to \pm\infty$, so ψ is strictly monotone decreasing for $t \to \pm\infty$, and again by (3.3.15) we derive

$$\lim_{t \to +\infty} \psi(t) = 0. \tag{3.3.17}$$

Now, we multiply (3.3.15) by $\dot{U}$ and find:

$$\ddot{\psi}\dot{U} = e^{2t}\psi(t) e^{U(t)} \left(2e^{U(t)} - 1\right) \dot{U}(t) = e^{2t}\psi(t) \frac{d}{dt} e^{U(t)} \left(e^{U(t)} - 1\right).$$

That is:

$$\begin{aligned}\frac{d}{dt}(\dot{\psi}\dot{U}) - \dot{\psi}\ddot{U} &= \frac{d}{dt}\left(e^{2t}\psi(t)e^{U(t)}\left(e^{U(t)}-1\right)\right) - \left(\frac{d}{dt}e^{2t}\psi(t)\right)e^{U(t)}\left(e^{U(t)}-1\right)\\ &= \frac{d}{dt}\left(e^{2t}\psi(t)e^{U(t)}\left(e^{U(t)}-1\right)\right) - 2e^{2t}\psi(t)e^{U(t)}\left(e^{U(t)}-1\right)\\ &\quad -e^{2t}e^{U(t)}\left(e^{U(t)}-1\right)\dot{\psi}.\end{aligned}$$

Therefore, if we use (3.3.10) we obtain:

$$\frac{d}{dt}\left[\dot{\psi}(t)\dot{U}(t) + e^{2t}\psi(t)e^{U(t)}\left(1-e^{U(t)}\right)\right] = 2e^{2t}\psi(t)e^{U(t)}\left(1-e^{U(t)}\right). \quad (3.3.18)$$

On the other hand, by (3.3.12), (3.3.14), (3.3.16), and (3.3.17) we have:

$$\lim_{t\to+\infty}\left(\dot{\psi}(t)\dot{U}(t) + e^{2t}\psi(t)e^{U(t)}\left(1-e^{U(t)}\right)\right) = 0. \quad (3.3.19)$$

Similarly, from (3.3.13) and (3.3.16) we obtain:

$$\lim_{t\to-\infty}\dot{\psi}(t)\dot{U}(t) = 0.$$

While (3.3.12) and (3.3.16) imply:

$$e^{2t}\psi(t)e^{U(t)}\left(1-e^{U(t)}\right) = O\left(te^{2(N+1)t}\frac{\psi(t)}{t}\right), \quad \text{as } t\to-\infty.$$

Thus,

$$\lim_{t\to-\infty}\left(\dot{\psi}(t)\dot{U}(t) + e^{2t}\psi(t)e^{U(t)}\left(1-e^{U(t)}\right)\right) = 0, \quad (3.3.20)$$

and together with (3.3.18), (3.3.19), and (3.3.20) we conclude

$$\int_{-\infty}^{+\infty} e^{2t}\psi(t)e^{U(t)}\left(1-e^{U(t)}\right)dt = 0,$$

in contradiction with the fact that, $e^{2t}\psi(t)e^{U(t)}(1-e^{U(t)}) > 0$ in $\mathbb{R}$. □

Remark 3.3.13 The non-degeneracy result of Theorem 3.3.12 was pointed out in [CFL] and also in [T7].

Next, we show how to use Theorem 3.3.12 in order to establish a uniqueness and non-degeneracy result for topological solution of (3.1.4).

For this purpose, recall that the topological boundary conditions (3.1.9) can be equivalently ensured by considering solutions of (3.1.4) in $L^1(\mathbb{R}^2)$ (see Proposition 3.2.4 and Remark 3.2.5).

For $R > 0$ set,

$$\mathcal{B}^1_R = \left\{u \in L^1(\mathbb{R}^2) : ||u||_{L^1(\mathbb{R}^2)} \le R\right\}$$

the ball of radius R in $L^1(\mathbb{R}^2)$. We have:

Theorem 3.3.14 *For every $R > 0$ there exist $\lambda_R > 0$ and $\mu_R > 0$ such that for $\lambda \geq \lambda_R$ any solution u of (3.1.4) in $\mathcal{B}^1_R$ satisfies:*

$$||\nabla\varphi||_2^2 + \lambda \int_{\mathbb{R}^2} e^u(2e^u - 1)\varphi^2 \geq \mu_R ||\varphi||^2_{H^1(\mathbb{R}^2)}, \ \forall \varphi \in H^1(\mathbb{R}^2).$$

In particular letting $u = u_0 + v$, then v defines a strict local minimum for I_λ in $H^1(\mathbb{R}^2)$.

Observe that Theorem 3.3.14 applies in particular to the *maximal* solution u_λ in Corollary 3.2.8. It turns out that u_λ is the *only* solution of (3.1.4) with bounded $\mathrm{L}^1(\mathbb{R}^2)$-norm uniformly with respect to λ.

In fact, from Theorem 3.3.14 we deduce the following:

Theorem 3.3.15 *If $\tilde{u}_\lambda$ is a solution for (3.1.4), (3.1.9) and $\tilde{u}_\lambda \neq u_\lambda$, then*

$$||\tilde{u}_\lambda||_{\mathrm{L}^1(\mathbb{R}^2)} \to +\infty, \ \textit{as } \lambda \to +\infty. \tag{3.3.21}$$

Proof of Theorem 3.3.15. Argue by contradiction, and suppose that besides the maximal solution $u_\lambda = u_0 + v_\lambda$ in Corollary 3.2.8 there exists a *second* solution $\tilde{u}_\lambda = u_0 + \tilde{v}_\lambda$ of (3.1.4) in $\mathcal{B}^1_R$ for $\lambda = \lambda_n \to +\infty$ and $R > 0$ sufficiently large. Observe that $v_{\lambda_n}(z) < \tilde{v}_{\lambda_n}(z)$ $\forall z \in \mathbb{R}^2$, and v_{λ_n} and $\tilde{v}_{\lambda_n}$ define two *strict* local minima for I_λ in $H^1(\mathbb{R}^2)$, as we deduce from Theorem 3.3.14.

Therefore, we are in position to use a "mountain-pass" construction for I_λ in the convex set $\{v \in H^1(\mathbb{R}^2) : v_{\lambda_n} \leq v \leq \tilde{v}_{\lambda_n}$ a.e.$\}$ and obtain another critical point $v^*_{\lambda_n}$ for I_{λ_n} which is *not* a local minimum and satisfies $v_{\lambda_n}(z) < v^*_{\lambda_n}(z) < \tilde{v}_{\lambda_n}(z)$ in $\mathbb{R}^2$ (see Theorem 12.8 in Chapter II of [St1]). Thus, $u_0 + v^*_{\lambda_n} \in \mathcal{B}^1_R$, which is impossible by virtue of Theorem 3.3.14. □

Remark 3.3.16 Recently K. Choe in [Cho] has used properties of radial Chern–Simons solutions derived in [CFL] to show that actually (3.3.21) can never occur, and thus obtained uniqueness for (3.1.4), (3.1.9) provided that $\lambda > 0$ is sufficiently large.

However, the question of uniqueness for problem (3.1.4), (3.1.9) for any fixed $\lambda > 0$ remains *open.*

Proof of Theorem 3.3.14. We argue by contradiction and suppose there exist sequences $\{\lambda_n\} \subset (1, +\infty)$ and $\{u_n\} \subset \mathrm{L}^1(\mathbb{R}^2)$ satisfying

$$-\Delta u_n = \lambda_n e^{u_n}(1 - e^{u_n}) - 4\pi \sum_{j=1}^{N} \delta_{z_j} \text{ in } \mathbb{R}^2, \tag{3.3.22}$$

in the sense of distributions,

$$\lim_{n\to+\infty} \lambda_n = +\infty \tag{3.3.23}$$

$$\lim_{n\to+\infty} ||u_n||_{\mathrm{L}^1(\mathbb{R}^2)} < +\infty \tag{3.3.24}$$

such that setting,

$$\mu_n = \inf\left\{ ||\nabla\varphi||_2^2 + \lambda_n \int_{\mathbb{R}^2} e^{u_n}(2e^{u_n} - 1)\varphi^2 \ \forall\varphi \in H^1(\mathbb{R}^2) \ ||\varphi||_{H^1(\mathbb{R}^2)} = 1 \right\} \tag{3.3.25}$$

we have

$$-\infty \leq \lim_{n\to\infty} \mu_n \leq 0. \tag{3.3.26}$$

We can use the analysis of the previous section for u_n to see in particular that

$$u_n < 0 \text{ in } \mathbb{R}^2 \text{ and } \lambda_n \int_{\mathbb{R}^2} e^{u_n}(1-e^{u_n}) + \lambda_n \int_{\mathbb{R}^2} (1-e^{u_n})^2 \leq c_0, \tag{3.3.27}$$

with $c_0 > 0$ a suitable constant (depending only on the multiplicity and the total number of the vortex points), and

$$||u_n||_{L^1(\mathbb{R}^2)} + \max_{\Omega_\delta}\{|\nabla u_n| + |u_n| + \lambda_n(1-e^{u_n})\} \to 0 \text{ as } n \to \infty, \tag{3.3.28}$$

for $\Omega_\delta = \mathbb{R}^2 \setminus \cup_{j=1}^N B_\delta(z_j)$ and $\delta > 0$.

Claim 1: For a suitable constant $c > 0$ and $\forall n \in \mathbb{N}$ we have,

$$\mu_n \geq -c. \tag{3.3.29}$$

To establish (3.3.29), take $\varphi \in H^1(\mathbb{R}^2)$ with $||\varphi||_{H^1(\mathbb{R}^2)} = 1$, and by means of (3.3.27) estimate:

$$\begin{aligned}
&\int_{\mathbb{R}^2} e^{u_n}(2e^{u_n}-1)\varphi^2 \\
&\quad = 2\int_{\mathbb{R}^2} e^{u_n}(e^{u_n}-1)\varphi^2 + \int_{\mathbb{R}^2}(e^{u_n}-1)\varphi^2 + \int_{\mathbb{R}^2}\varphi^2 \\
&\quad \geq -\left(2\left(\int_{\mathbb{R}^2} e^{2u_n}(e^{u_n}-1)^2\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^2}(e^{u_n}-1)^2\right)^{\frac{1}{2}}\right)\left(\int_{\mathbb{R}^2}\varphi^4\right)^{\frac{1}{2}} + \int_{\mathbb{R}^2}\varphi^2 \\
&\quad \geq -\left(2\left(\int_{\mathbb{R}^2} e^{u_n}(1-e^{u_n})\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^2}(1-e^{u_n})^2\right)^{\frac{1}{2}}\right)\left(\int_{\mathbb{R}^2}\varphi^4\right)^{\frac{1}{2}} + \int_{\mathbb{R}^2}\varphi^2 \\
&\quad \geq -3\left(\frac{c_0}{\lambda_n}\int_{\mathbb{R}^2}\varphi^4\right)^{\frac{1}{2}} + \int_{\mathbb{R}^2}\varphi^2.
\end{aligned}$$

By the Sobolev inequality

$$\left(\int_{\mathbb{R}^2}\varphi^4\right)^{\frac{1}{2}} \leq 2||\varphi||_{L^2(\mathbb{R}^2)}||\nabla\varphi||_{L^2(\mathbb{R}^2)} \leq 2||\varphi||_{L^2(\mathbb{R}^2)},$$

we deduce

$$\begin{aligned}
&||\nabla\varphi||^2_{L^2(\mathbb{R}^2)} + \lambda_n \int_{\mathbb{R}^2} e^{u_n}(2e^{u_n}-1)\varphi^2 \\
&\quad \geq 1 + \left(1-\frac{1}{\lambda_n}\right)\left(\lambda_n\int_{\mathbb{R}^2}\varphi^2\right) - 6\sqrt{c_0}\left(\lambda_n\int_{\mathbb{R}^2}\varphi^2\right)^{\frac{1}{2}} \\
&\quad \geq 1 - \frac{9c_0}{1-\frac{1}{\lambda_n}} \geq -c,
\end{aligned} \tag{3.3.30}$$

for suitable $c > 0$, and sufficiently large n. Therefore, (3.3.29) holds.

As a consequence of (3.3.29), (3.3.26) we set:

$$\mu_0 := \lim_{n\to\infty} \mu_n \leq 0. \tag{3.3.31}$$

Furthermore by the given properties of u_n, we see that the extremal problem (3.3.25) attains its infimum at a function $\varphi_n \in H^1(\mathbb{R}^2)$ satisfying:

$$\begin{cases} -\Delta\varphi_n = \dfrac{\lambda_n}{1-\mu_n} e^{u_n}(1-2e^{u_n})\varphi_n + \dfrac{\mu_n}{1-\mu_n}\varphi_n \text{ in } \mathbb{R}^2 \\ \varphi_n > 0 \text{ in } \mathbb{R}^2, \ ||\varphi_n||_{H^1(\mathbb{R}^2)} = 1. \end{cases} \tag{3.3.32}$$

Using (3.3.30) with $\varphi = \varphi_n$ we find:

$$\begin{aligned} \mu_n = ||\nabla\varphi_n||^2_{L^2} + \lambda_n \textstyle\int_{\mathbb{R}^2} e^{u_n}(2e^{u_n}-1)\varphi_n^2 \geq 1 - \left(1 - \frac{1}{\lambda_n}\right)\left(\lambda_n \int_{\mathbb{R}^2}\varphi_n^2\right) \\ -6\sqrt{c_0}\left(\lambda_n \textstyle\int_{\mathbb{R}^2}\varphi_n^2\right)^{\frac{1}{2}}. \end{aligned}$$

So, by means of (3.3.31) we deduce that

$$\lambda_n \int_{\mathbb{R}^2} \varphi_n^2 \leq A, \tag{3.3.33}$$

$\forall n \in \mathbb{N}$ and suitable $A > 0$.

Claim 2: There exist $j_0 \in \{1, \ldots, N\}$ and $r_0 > 0$ such that, $\forall \rho \in (0, r_0]$, we find a constant $a_\rho > 0$:

$$\lambda_n \int_{B_\rho(z_{j_0})} \varphi_n^2 \geq a_\rho. \tag{3.3.34}$$

To obtain (3.3.34), we argue by contradiction and suppose there exists $\delta > 0$ sufficiently small such that

$$\lambda_n \int_{\cup_{j=1}^N B_\delta(z_j)} \varphi_n^2 \to 0, \text{ as } n \to \infty. \tag{3.3.35}$$

As a consequence of (3.3.35) and (3.3.28) we have:

$$\begin{aligned} &\left|\lambda_n \int_{\mathbb{R}^2} e^{u_n}(2e^{u_n}-1)\varphi_n^2 - \lambda_n \int_{\mathbb{R}^2}\varphi_n^2\right| \\ &\quad = \left|2\lambda_n \int_{\mathbb{R}^2} e^{u_n}(e^{u_n}-1)\varphi_n^2 + \lambda_n \int_{\mathbb{R}^2}(e^{u_n}-1)\varphi_n^2\right| \\ &\quad \leq 2\lambda_n \int_{\Omega_\delta} e^{u_n}(1-e^{u_n})\varphi_n^2 + \lambda_n \int_{\Omega_\delta}(1-e^{u_n})\varphi_n^2 + 3\lambda_n \int_{\cup_{j=1}^N B_\delta(z_j)} \varphi_n^2 \\ &\quad \leq 3\left(\left(\sup_{\Omega_\delta} \lambda_n(1-e^{u_n})\right)||\varphi_n||^2_{L^2} + \lambda_n \int_{\cup_{j=1}^N B_\delta(z_j)} \varphi_n^2\right) \\ &\quad \leq 3\left(\sup_{\Omega_\delta} \lambda_n(1-e^{u_n}) + \lambda_n \int_{\cup_{j=1}^N B_\delta(z_j)} \varphi_n^2\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, as $n \to \infty$,

$$\mu_n = ||\nabla \varphi_n||^2_{L^2(\mathbb{R}^2)} + \lambda_n \int_{\mathbb{R}^2} \varphi_n^2 + o(1) \geq 1 + o(1),$$

in contradiction with (3.3.31), and (3.3.35) is established.

By replacing $u_n(z)$ with $u_n(z + z_{j_0})$, we can always suppose that $z_{j_0} = 0$ and denote by $\nu \in \mathbb{N}$ the corresponding multiplicity. Furthermore, by taking $r_0 > 0$ smaller if necessary, we can assume that the origin is the only vortex point in $B_{r_0}(0)$. Thus, u_n satisfies

$$-\Delta u_n = \lambda_n e^{u_n}(1 - e^{u_n}) - 4\pi \nu \delta_{z=0} \text{ in } B_{r_0} \tag{3.3.36}$$

$$\lambda_n \int_{B_{r_0}} (1 - e^{u_n}) \leq c_0 \text{ and } u_n < 0 \text{ in } B_{r_0}, \tag{3.3.37}$$

for suitable $c_0 > 0$. In addition, $\forall \rho \in (0, r_0)$ and from (3.3.28), we know that

$$\max_{\rho \leq |z| \leq r_0} \left(|u_n| + \lambda_n(1 - e^{u_n})\right) \to 0 \tag{3.3.38}$$

$$\lambda_n \int_{B_{r_0}(0)} (1 - e^{u_n})^2 = 4\pi \nu^2 + o(1), \tag{3.3.39}$$

as $n \to \infty$. We carry out a blow-up analysis, and consider the scaled function:

$$\hat{u}_n(z) = u_n\left(\frac{z}{\sqrt{\lambda_n}}\right) \quad z \in D_n := B_{\sqrt{\lambda_n} r_0} \tag{3.3.40}$$

$$\hat{\varphi}_n(z) = \varphi_n\left(\frac{z}{\sqrt{\lambda_n}}\right) \quad z \in D_n. \tag{3.3.41}$$

We decompose

$$\hat{u}_n(z) = 2\nu \log|z| + \hat{v}_n(z), \; z \in D_n, \tag{3.3.42}$$

with $\hat{v}_n$ satisfying

$$-\Delta \hat{v}_n = |z|^{2\nu} e^{\hat{v}_n}(1 - |z|^{2\nu} e^{\hat{v}_n}) \text{ in } D_n \tag{3.3.43}$$

$$\int_{D_n} (1 - |z|^{2\nu} e^{\hat{v}_n}) \leq c_0. \tag{3.3.44}$$

Claim 3: The following holds:

(a) $\hat{v}_n$ is uniformly bounded in $C^{2,\alpha}$-topology; (3.3.45)

(b) $\forall \delta > 0$, there exists $c_\delta > 0 : \sup_{D_n \setminus B_\delta(0)} |\hat{u}_n| \leq c_\delta$; (3.3.46)

(c) $\forall p \geq 1$, there exists $C_p > 0 : ||\hat{u}_n||_{L^p(\mathbb{R}^2)} \leq C_p$. (3.3.47)

To establish (a), we start by showing that $\forall R > 0$ there exists $C_R > 0$ such that

$$\inf_{B_R} \hat{v}_n \geq -C_R. \tag{3.3.48}$$

To this purpose, suppose by contradiction that for any $R > 0$ sufficiently large,

$$\inf_{B_R} \hat{v}_n \to -\infty,$$

as $n \to +\infty$ (possibly along a subsequence). Set $f_n = e^{\hat{u}_n}(1 - e^{\hat{u}_n})$. Then $0 \leq f_n < 1$ in $\mathbb{R}^2$ and,

$$\begin{cases} -\Delta \hat{v}_n = f_n \text{ in } B_{2R} \\ \hat{v}_n|_{\partial B_{2R}} \leq 2\nu \log \frac{1}{2R}. \end{cases}$$

Hence, we can use the Harnack inequality (see Proposition 5.2.8) to obtain constants $\gamma \in (0, 1)$ and $C > 0$ (independent of n) such that

$$\sup_{B_R} \hat{v}_n \leq \gamma \inf_{B_R} \hat{v}_n + C.$$

Therefore, $\sup_{B_R} \hat{v}_n \to -\infty$ and $\int_{B_R}(1 - |x|^{2\nu} e^{\hat{v}_n}) = \pi R^2 + o(1)$ as $n \to \infty$. But for R sufficiently large this contradicts (3.3.44), and (3.3.48) follows. Hence $\hat{v}_n$ as well as $\Delta \hat{v}_n$ are uniformly bounded in $L^2_{\text{loc}}(\mathbb{R}^2)$, and we can use well-known elliptic estimates together with a bootstrap argument to obtain a uniform bound for $\hat{v}_n$ in $C^{2,\alpha}_{\text{loc}}$ -norm as claimed.

To establish (b), again by contradiction, suppose there exists

$$z_n \in D_n \setminus B_\delta(0) : \ \hat{u}_n(z_n) \to -\infty.$$

Hence $\hat{v}_n(z_n) \to -\infty$, and by (3.3.48) we see that necessarily $|z_n| \to +\infty$, as $n \to +\infty$. Therefore for any $R > 0$,

$$B_{2R}(z_n) \subset D_n \setminus B_\delta(0),$$

provided that n is sufficiently large. As above, we can use the Harnack inequality for $\tilde{u}_n(z) = \hat{u}_n(z_n + z)$, since it satisfies:

$$\begin{cases} -\Delta \tilde{u}_n = f_n(z_n + x) \text{ in } B_{2R}(0) \\ \tilde{u}_n|_{\partial B_{2R}} < 0, \ \tilde{u}_n(0) \to -\infty, \ \text{as } n \to \infty. \end{cases}$$

In this way, we deduce that $\sup_{B_R} \tilde{u}_n \to -\infty$ and

$$\int_{B_R(z_n)} (1 - e^{\hat{u}_n}) = \int_{B_R(0)} (1 - e^{\tilde{u}_n}) = \pi R^2 + o(1) \text{ as } n \to \infty,$$

for every $R > 0$, in contradiction with (3.3.44).

Finally, combining (a) and (b), we see that $\forall p \geq 1$,

$$||\hat{u}_n||_{L^p(B_R)} \leq C_{p,R}$$

for a suitable constant $C_{p,R} > 0$, depending only on p and R. While for every $\delta > 0$, we estimate

$$\int_{\{|z|\geq\delta\}} |\hat{u}_n| \leq (1 + \sup_{D_n \setminus B_\delta(0)} |\hat{u}_n|) \int_{\mathbb{R}^2} (1 - e^{\hat{u}_n}) \leq C_\delta,$$

for suitable $C_\delta > 0$. By the above estimates (c) easily follows. □

By virtue of (3.3.45), we can use a diagonalization process to obtain a subsequence of $\hat{v}_n$ (denoted in the same way) such that

$$\hat{v}_n \to v \text{ uniformly in } C^2_{\text{loc}}; \tag{3.3.49}$$

and for

$$u(z) = 2\nu \log |z| + v(z),$$

we have

$$\begin{cases} -\Delta u = e^u(1 - e^u) - 4\pi\nu\delta_{z=0}, \text{ in } \mathbb{R}^2 \\ u < 0, \ u \in \mathrm{L}^p(\mathbb{R}^2), \ p \geq 1 \\ \int_{\mathbb{R}^2}(1 - e^u) < +\infty. \end{cases}$$

In other words, u coincides with the unique radially symmetric solution of problem (3.3.1), for which properties (3.3.2), (3.3.3), and (3.3.4) hold as well as Theorem 3.3.12.

Claim 4:

$$||\hat{u}_n - u||_{\mathrm{L}^2(D_n)} + \sup_{D_n} |\hat{u}_n - u| \to 0, \text{ as } n \to +\infty. \tag{3.3.50}$$

To establish (3.3.50), we observe that $\hat{u}_n - u = \hat{v}_n - v$, and so

$$||\hat{u}_n - u||_{C^2(B_R)} \to 0 \text{ as } n \to +\infty, \tag{3.3.51}$$

for every $R > 0$.

On the other hand, since $\hat{u}_n - u$ is uniformly bounded in $D_n \setminus B_\delta(0)$ (see (3.3.46)), from (3.2.24) we deduce the estimate

$$\int_{D_n \setminus B_\delta(0)} |\hat{u}_n - u|^2 \leq (1 + C_\delta)^2 ||e^{-2u}||_{\mathrm{L}^\infty(\mathbb{R}^2 \setminus B_\delta(0))} \int_{D_n} (e^{\hat{u}_n} - e^u)^2, \tag{3.3.52}$$

with suitable $C_\delta > 0$.

But,

$$\int_{D_n} (e^{\hat{u}_n} - e^u)^2 = \int_{D_n} (e^{\hat{u}_n} - 1)^2 + \int_{D_n} (e^u - 1)^2 - 2\int_{D_n} (1 - e^{\hat{u}_n})(1 - e^u)$$
$$= 4\pi\nu^2 + 4\pi\nu^2 - 2\int_{D_n} (1 - e^{\hat{u}_n})(1 - e^u) + o(1) \to 0 \text{ as } n \to +\infty,$$

since, by dominated convergence, we have: $\int_{D_n}(1-e^{\hat{u}_n})(1-e^u) \to \int_{\mathbb{R}^2}(1-e^u)^2 = 4\pi\nu^2$, as $n \to \infty$.

Hence,

$$||e^{\hat{u}_n} - e^u||_{L^2(D_n)} \to 0 \text{ as } n \to +\infty,$$

and by (3.3.51) and (3.3.52), we conclude that $||\hat{u}_n - u||_{L^2(D_n)} \to 0$, as $n \to \infty$.

Furthermore,

$$||e^{\hat{u}_n}(1-e^{\hat{u}_n}) - e^u(1-e^u)||_{L^2(D_n)} \le 2||e^{\hat{u}_n} - e^u||_{L^2(D_n)} \to 0, \text{ as } n \to +\infty.$$

Thus, we have established that

$$||\hat{u}_n - u||_{L^2(D_n)} + ||\Delta(\hat{u}_n - u)||_{L^2(D_n)} \to 0, \text{ as } n \to +\infty.$$

Therefore, by well-known elliptic estimates, for every $\rho \in (0, r_0)$ and $z \in B_{\rho\sqrt{\lambda_n}}$, we see that $B_2(z) \subset D_n$ (for n large) and

$$\begin{aligned}\sup_{B_1(z)} |\hat{u}_n - u| &\le C\big(||\hat{u}_n - u||_{L^2(B_2(z))} + ||\Delta(\hat{u}_n - u)||_{L^2(B_2(z))}\big)\\ &\le C\big(||\hat{u}_n - u||_{L^2(D_n)} + ||\Delta(\hat{u}_n - u)||_{L^2(D_n)}\big)\end{aligned}$$

with a suitable constant $C > 0$ independent of z and n. Consequently, $\forall\, \rho \in (0, r_0)$, we find: $\sup_{|z|\le\sqrt{\lambda_n}\rho} |\hat{u}_n - u| \to 0$, as $n \to +\infty$. Also, using (3.3.28) and (3.3.3), we see that

$$\begin{aligned}\sup_{\sqrt{\lambda_n}\rho\le|z|\le r_0\sqrt{\lambda_n}} |\hat{u}_n - u| &\le \sup_{\sqrt{\lambda_n}\rho\le|z|\le r_0\sqrt{\lambda_n}} |\hat{u}_n| + \sup_{|z|\ge\sqrt{\lambda_n}\rho} |u|\\ &\le \sup_{\rho\le|z|\le r_0} |u_n| + C_\varepsilon e^{-(1-\varepsilon)\rho\sqrt{\lambda_n}} \to 0\end{aligned}$$

as $n \to \infty$, and so (3.3.50) is established.

Observe that, as a consequence of (3.3.51), we also have:

$$\sup_{D_n}|e^{\hat{u}_n}(1-e^{\hat{u}_n}) - e^u(1-e^u)| + \sup_{D_n}|e^{\hat{u}_n} - e^u| \to 0, \text{ as } n \to +\infty. \tag{3.3.53}$$

Next, we turn to analyze the asymptotic behavior of $\hat{\varphi}_n = \varphi_n\left(\frac{z}{\sqrt{\lambda_n}}\right)$ which we know to satisfy:

$$\begin{cases}-\Delta\hat{\varphi}_n = \frac{1}{1-\mu_n}e^{\hat{u}_n}(1-2e^{\hat{u}_n})\hat{\varphi}_n + \frac{\mu_n}{\lambda_n(1-\mu_n)}\hat{\varphi}_n \text{ in } D_n\\ \hat{\varphi}_n > 0.\end{cases} \tag{3.3.54}$$

Recalling (3.3.33), we derive the following uniform estimate,

$$||\nabla\hat{\varphi}_n||^2_{L^2(D_n)} + ||\hat{\varphi}_n||^2_{L^2(D_n)} = ||\nabla\varphi_n||_{L^2(B_{r_0})} + \lambda_n\int_{B_{r_0}}\varphi_n^2 \le C$$

for suitable $C > 0$.

Hence, using once more elliptic estimates, we see that $\hat{\varphi}_n$ is uniformly bounded in $C^{2,\alpha}_{\mathrm{loc}}$-topology and, by passing to a subsequence if necessary (denoted in the same way), we can assume that

$$\hat{\varphi}_n \to \varphi \text{ in } C^2_{\mathrm{loc}}.$$

Furthermore, φ satisfies:

$$\begin{cases} -\Delta\varphi = \frac{1}{1+|\mu_0|} e^u(1-2e^u)\varphi \text{ in } \mathbb{R}^2 \\ \varphi \in H^1(\mathbb{R}^2), \ \varphi \geq 0. \end{cases}$$

By virtue of (3.3.6), we see that $\mu_0 = 0$, and so we can use Theorem 3.3.12 to conclude that $\varphi = 0$.

On the other hand, using the cut-off function $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi = 1$ in B_1, $\chi = 0$ in $\mathbb{R}^2 \setminus B_2$ and $0 \leq \chi \leq 1$, for $\rho \in \left(0, \frac{r_0}{2}\right)$ we define:

$$\chi_n(z) = \chi\left(\frac{z}{\rho\sqrt{\lambda_n}}\right).$$

Then we find,

$$\begin{aligned} \int_{D_n} (-\Delta\hat{\varphi}_n)(\chi_n\hat{\varphi}_n) &= \frac{1}{1-\mu_n}\int_{D_n} e^{\hat{u}_n}(1-2e^{\hat{u}_n})\chi_n\hat{\varphi}_n^2 + \frac{\mu_n}{\lambda_n(1-\mu_n)}\int_{D_n}\chi_n\hat{\varphi}_n^2 \\ &= \frac{2}{1-\mu_n}\int_{D_n} e^{\hat{u}_n}(1-e^{\hat{u}_n})\chi_n\hat{\varphi}_n^2 + \frac{1}{1-\mu_n}\int_{D_n}(1-e^{\hat{u}_n})\chi_n\hat{\varphi}_n^2 \\ &\quad -\frac{1}{1-\mu_n}\int_{D_n}\chi_n\hat{\varphi}_n^2 + o(1) \\ &= \left(2\int_{D_n} e^u(1-e^u)\chi_n\hat{\varphi}_n^2 + \int_{D_n}(1-e^u)\chi_n\hat{\varphi}_n^2\right) - \int_{D_n}\chi_n\hat{\varphi}_n^2 + o(1). \end{aligned}$$

By the exponential decay estimate in (3.3.3), we get

$$\int_{D_n}\left(2e^u(1-e^u) + (1-e^u)\right)\chi_n\hat{\varphi}_n^2 \leq 3\int_{B_R}\hat{\varphi}_n^2 + C_\varepsilon e^{-(1-\varepsilon)R},$$

for every $R > 0$ and with C_ε depending on $\varepsilon \in (0,1)$ only. Since $\hat{\varphi}_n^2 \to 0$ uniformly in C^2_{loc}, from the estimate above we deduce that

$$\int_{D_n}\left(2e^u(1-e^u) + (1-e^u)\right)\chi_n\hat{\varphi}_n^2 \to 0, \text{ as } n \to \infty.$$

On the other hand, observing that

$$\begin{aligned} \int_{D_n}(\Delta\hat{\varphi}_n)\chi_n\hat{\varphi}_n &= \int_{D_n}\chi_n|\nabla\hat{\varphi}_n|^2 - \frac{1}{2}\int_{D_n}\nabla\chi_n\cdot\nabla\hat{\varphi}_n^2 \\ &= \int_{D_n}\chi_n|\nabla\hat{\varphi}_n|^2 + \frac{1}{2}\int_{D_n}\Delta\chi_n\cdot\hat{\varphi}_n^2 \\ &\geq \int_{D_n}\chi_n|\nabla\hat{\varphi}_n|^2 - \frac{||\Delta\chi||_{L^\infty}}{2\rho^2\lambda_n}\int_{B_{r_0}}\hat{\varphi}_n^2 \geq -\frac{C}{\lambda_n} \end{aligned}$$

with a suitable constant $C > 0$, we can combine the estimates above to deduce that $\int_{D_n} \chi_n \hat{\varphi}_n^2 \to 0$, as $n \to +\infty$.

Consequently,

$$\lambda_n \int_{B_\rho} \varphi_n^2 = \int_{\{|z| \le \rho\sqrt{\lambda_n}\}} \hat{\varphi}_n^2 \le \int_{D_n} \chi_n \hat{\varphi}_n^2 \to 0, \text{ as } n \to +\infty,$$

and recalling (3.3.34), we arrive at a contradiction and conclude the proof of Theorem 3.3.14. □

We mention that Theorem 3.3.14 and Theorem 3.3.15 are the results of ideas recently introduced by the author in [T7] to establish uniqueness of topological-type solutions in the periodic case.

3.4 Planar non-topological Chern–Simons vortices

In this section we treat "non-topological" Chern–Simons vortices, namely, solutions to (3.1.4), (3.1.5), and (3.1.8) subject to the boundary condition (3.1.10). We present the perturbative approach of Chae–Imanuvilov [ChI1], successfully used to handle other Chern–Simons models (cf. [ChI2], [ChI3], and [Ch3]), electroweak vortices and strings (cf. [ChT1] and [ChT2]), as well as cosmic strings (see [Ch1], [Ch4], and [ChCh1]). See also [Ch2] for an application of such an approach in the context of the Born–Infield theory.

To be more specific, we focus as above (without loss of generality) to the case $\nu = 1$.

The approach introduced by Chae–Imanuvilov in [ChI1] to obtain "non-topological" Chern–Simons vortices is based on the observation that if we have a solution u for (3.1.4), then for every $\varepsilon > 0$, the scaled function

$$u_\varepsilon(z) = u\left(\frac{z}{\varepsilon}\right) + 2\log\left(\frac{1}{\varepsilon}\right)$$

satisfies

$$-\Delta u_\varepsilon = \lambda e^{u_\varepsilon} - \lambda\varepsilon^2 e^{2u_\varepsilon} - 4\pi \sum_{j=1}^{N} \delta_{\varepsilon z_j} \text{ in } \mathbb{R}^2. \tag{3.4.1}$$

Therefore, we may regard the ε-scaled problem (3.4.1) as a perturbation of the "singular" Liouville problem:

$$\begin{cases} -\Delta u = \lambda e^u - 4\pi \sum_{j=1}^{N} \delta_{\varepsilon z_j} \\ \int_{\mathbb{R}^2} e^u < \infty. \end{cases} \tag{3.4.2}$$

This observation already puts an emphasis on the role played by Liouville equations in the search for non-topological solutions. This fact will be further exploited for the periodic case and in the electroweak model.

Observe that by Liouville formula (2.2.3), we can exhibit an explicit solution for (3.4.2). Indeed, let

$$f(z) = (N+1)\prod_{j=1}^{N}(z - z_j) \text{ and } F(z) = \int_0^z f(\xi)\,d\xi,$$

and set

$$f_\varepsilon(z) = (N+1)\prod_{j=1}^{N}(z - \varepsilon z_j) \text{ and } F_\varepsilon(z) = \int_0^z f_\varepsilon(\xi)\,d\xi.$$

By (2.2.3) we know that

$$u^0_{\varepsilon,a}(z) = \log \frac{8|f_\varepsilon(z)|^2}{\lambda(1+|F_\varepsilon(z)+a|^2)^2} \tag{3.4.3}$$

satisfies (3.4.2) for any $\varepsilon \in \mathbb{R}$ and $a \in \mathbb{C}$. Incidentally, notice that all solutions of (3.4.2) are obtained in this way (cf. [PT] and see [CW], [CL1], [CL2], and [CK1]). Thus, we can reasonably search for the solution of our problem

$$\begin{cases} -\Delta u = \lambda e^u(1-e^u) - 4\pi \displaystyle\sum_{j=1}^{N} \delta_{\varepsilon z_j} \\ u(z) \to -\infty \text{ as } |z| \to +\infty \\ e^u(1-e^u) \in \mathrm{L}^1(\mathbb{R}^2) \end{cases} \tag{3.4.4}$$

in the form

$$u(z) = u^0_{\varepsilon,a}(\varepsilon z) + \log \varepsilon^2 + \varepsilon^2 w(\varepsilon z), \tag{3.4.5}$$

with w a suitable function that identifies the error term and satisfies

$$-\Delta w = \lambda e^{u^0_{\varepsilon,a}} \left(\frac{e^{\varepsilon^2 w} - 1}{\varepsilon^2} \right) - \lambda e^{2(u^0_{\varepsilon,a} + \varepsilon^2 w)}. \tag{3.4.6}$$

We consider the free parameters ε and a as part of our unknowns and concentrate around the values $\varepsilon = 0$ and $a = 0$ where (3.4.6) reduces to

$$\Delta w + \rho w = \frac{1}{\lambda}\rho^2 \tag{3.4.7}$$

with $\rho = \lambda e^{u^0_{\varepsilon=0,a=0}}$, the radial function, given as follows:

$$\rho(r) = \frac{8(N+1)^2 r^{2N}}{\left(1 + r^{2(N+1)}\right)^2}, \quad r = |z|. \tag{3.4.8}$$

Remark 3.4.17 For later use, notice that the definition of ρ, as well as (3.4.7), make good sense also for $N = 0$.

We can analyze (3.4.7) within the class of radial functions, where we find an explicit solution $w_0 = w_0(r)$ given by the expression (3.4.30) below.

Thus, using the decomposition

$$w(z) = w_0(|z|) + u_1(z), \tag{3.4.9}$$

we need to solve for u_1 in the following equation:

$$\begin{aligned} P(u_1, a, \varepsilon) := \Delta u_1 + \lambda e^{u^0_{\varepsilon,a}} \left(\frac{e^{\varepsilon^2 w_0 + \varepsilon^2 u_1} - 1}{\varepsilon^2} \right) - \rho w_0 \\ + \lambda e^{2(u^0_{\varepsilon,a} + \varepsilon^2 w_0 + \varepsilon^2 u_1)} - \frac{1}{\lambda} \rho^2 = 0. \end{aligned} \tag{3.4.10}$$

To attain this goal, we aim to apply the Implicit Function Theorem (cf. [Nir]) to the operator P considered on suitable functional spaces where it extends smoothly at $\varepsilon = 0$ to satisfy $P(0, 0, 0) = 0$. To this purpose, Chae–Imanuvilov in [ChI1] have introduced the spaces

$$\begin{aligned} X_\alpha &= \left\{ u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2) : \ (1 + |z|^{2+\alpha}) u^2 \in \mathrm{L}^1(\mathbb{R}^2) \right\}, \ \alpha > 0 \\ Y_\alpha &= \left\{ u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2) : \ \Delta u \in X_\alpha, \ \frac{u}{(1+|z|)^{1+\frac{\alpha}{2}}} \in \mathrm{L}^2(\mathbb{R}^2) \right\}, \ \alpha > 0 \end{aligned} \tag{3.4.11}$$

equipped respectively with the scalar product

$$(u, v)_{X_\alpha} = \int_{\mathbb{R}^2} (1 + |z|^{2+\alpha}) uv \ \text{ and } \ (u, v)_{Y_\alpha} = (\Delta u, \Delta v)_{X_\alpha} + \int_{\mathbb{R}^2} \frac{uv}{(1 + |z|^{2+\alpha})},$$

and relative norms denoted by $|| \cdot ||_{X_\alpha}$ and $|| \cdot ||_{Y_\alpha}$, respectively.

For any $\alpha > 0$, the following continuous embedding properties hold:

$$\begin{aligned} X_\alpha &\hookrightarrow \mathrm{L}^q(\mathbb{R}^2), \ \forall q \in [1, 2); \\ Y_\alpha &\hookrightarrow C^0_{\mathrm{loc}}(\mathbb{R}^2). \end{aligned}$$

Furthermore, we have:

Lemma 3.4.18 *Let $\alpha \in (0, 1)$ and $v \in Y_\alpha$.*
(a) If v is harmonic, then v is a constant.
(b) The following estimates hold:

$$|v(z)| \leq C ||v||_{Y_\alpha} \log(1 + |z|), \ \textit{in } \mathbb{R}^2; \tag{3.4.12}$$

$$||\nabla v||_{\mathrm{L}^p} \leq C_p ||v||_{Y_\alpha}, \ \textit{for every } p > 2. \tag{3.4.13}$$

where $C > 0$ and $C_p > 0$ are suitable constants depending on α and (α, p) respectively.

Proof. To establish (a), we first observe that by standard elliptic regularity, any harmonic function υ in Y_α is smooth. Moreover, if we express υ according to its Fourier decomposition

$$\upsilon(z) = \sum_{k=-\infty}^{+\infty} \xi_k(r) e^{i\theta k},\ z = re^{i\theta},$$

with $\xi_k = \xi_k(r) \in \mathbb{C}$ such that $\xi_{-k} = \overline{\xi}_k$ and satisfies

$$\ddot{\phi} + \frac{1}{r}\dot{\phi} - \frac{k^2}{r^2}\phi = 0,\ \forall\, k \in \mathbb{Z}^+. \tag{3.4.14}$$

Note that for $k \geq 1$, the functions $\phi_{1,k} = r^k$ and $\phi_{2,k} = \frac{1}{r^k}$ represent a fundamental set of solutions to (3.4.14), while for $k = 0$ we have $\phi_{1,0} = 1$ and $\phi_{2,0}(r) = \log(r)$ as fundamental solutions. The smoothess of υ in $\mathbb{R}^2$ and the fact that $\frac{\upsilon}{1+r^{1+\alpha}} \in \mathrm{L}^2(\mathbb{R}^2)$ for $\alpha \in (0, 1)$ imply that $\xi_k \equiv 0$ for every $k \in \mathbb{N}$, and $\xi_0 =$ constant, so that (a) follows. To obtain (b), let $\upsilon \in Y_\alpha$ so that $\Delta \upsilon := g \in X_\alpha$. Set

$$\tilde{\upsilon}(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|z-\eta| g(\eta) d\,\eta, \tag{3.4.15}$$

then $\Delta\tilde{\upsilon} = \Delta\upsilon$ in $\mathbb{R}^2$. We estimate:

$$\begin{aligned}
|\tilde{\upsilon}(z)| &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} (1+|\eta|^{2+\alpha})^{\frac{1}{2}} |g(\eta)| \frac{|\log|z-\eta||}{(1+|\eta|^{2+\alpha})^{\frac{1}{2}}} \\
&\leq \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} (1+|\eta|^{2+\alpha}) g^2(\eta) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \frac{\log^2|z-\eta|}{(1+|\eta|^{2+\alpha})} \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2\pi} ||\upsilon||_{Y_\alpha} \left(\int_{\mathbb{R}^2} \frac{\log^2|z-\eta|}{(1+|\eta|^{2+\alpha})} \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.4.16}$$

In turn, for $|z| > 1$ the integral above can be estimated as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\log^2|z-\eta|}{(1+|\eta|^{2+\alpha})} \\
&\quad \leq \int_{\{|z-\eta|<1\}} \log^2\left(\frac{1}{|z-\eta|}\right) + \int_{\{1<|z-\eta|\leq 2|z|\}} \frac{\log^2|z-\eta|}{(1+|\eta|^{2+\alpha})} \\
&\qquad + \int_{\{|z-\eta|>2|z|\}} \frac{\log^2|z-\eta|}{(1+|\eta|^{2+\alpha})} \\
&\quad \leq 2\pi \int_0^1 \log^2\left(\frac{1}{r}\right) r\,dr + \log^2(2|z|) \int_{\mathbb{R}^2} \frac{1}{1+|\eta|^{2+\alpha}} \\
&\qquad + \int_{\{|\eta|\geq|z|\}} \frac{\log^2\left(|\eta|\left(1+\frac{|z|}{|\eta|}\right)\right)}{(1+|\eta|^{2+\alpha})} \\
&\quad \leq \log^2|z| \int_{\mathbb{R}^2} \frac{1}{1+|\eta|^{2+\alpha}} + \int_{\mathbb{R}^2} \frac{\log^2(2|\eta|)}{(1+|\eta|^{2+\alpha})} + C.
\end{aligned} \tag{3.4.17}$$

Thus, by combining (3.4.16) with (3.4.17) we conclude:

$$|\tilde{v}(z)| \leq C_\alpha ||v||_{Y_\alpha} \log(|z|+1). \tag{3.4.18}$$

By differentiating (3.4.15) with respect to z and using well-known potential estimates (cf. [GT]), we get that, for every $q \in (1,2)$,

$$||\nabla \tilde{v}||_{L^{\frac{2q}{2-q}}(\mathbb{R}^2)} \leq C||\Delta \tilde{v}||_{L^q(\mathbb{R}^2)} \leq C||\Delta v||_{X_\alpha} \leq C||v||_{Y_\alpha} \tag{3.4.19}$$

for a suitable constant $C > 0$ depending on α and q. In particular, from (3.4.18) it follows that, $\tilde{v} \in Y_\alpha$ and $||\tilde{v}||_{Y_\alpha} \leq C_\alpha ||v||_{Y_\alpha}$ for a suitable constant $C_\alpha > 0$. So $v - \tilde{v}$ defines a harmonic function in Y_α, and for $\alpha \in (0,1)$, we can use part (a) to conclude that

$$c = v - \tilde{v} \tag{3.4.20}$$

for a suitable constant $c \in \mathbb{R}$, with

$$||c||_{Y_\alpha} \leq ||v||_{Y_\alpha} + ||\tilde{v}||_{Y_\alpha} \leq C_\alpha ||v||_{Y_\alpha}. \tag{3.4.21}$$

Hence, it suffices to combine (3.4.18), (3.4.19), (3.4.20), and (3.4.21) to obtain the desired conclusion. □

Working with the spaces X_α and Y_α is particularly advantageous for the linear operator

$$L = \Delta + \rho : Y_\alpha \longrightarrow X_\alpha, \tag{3.4.22}$$

as we can characterize explicitly $KerL \subset Y_\alpha$ and $\mathrm{Im}L \subset X_\alpha$.

To this purpose, consider the family of functions:

$$U_{\mu,a}(z) = u^0_{\varepsilon=0,a}(\mu z) + \log \mu^2, \ \mu > 0, \ \text{and } a \in \mathbb{C};$$

satisfying:

$$-\Delta U = \lambda e^U - 4\pi N \delta_{z=0}. \tag{3.4.23}$$

Letting $u^0 = U_{\mu=1,a=0} = \log \rho$ and using polar coordinates, we see that the following functions belong to $KerL$ in Y_α, $\forall \alpha > 0$:

$$\begin{aligned} \phi_0 &= \frac{1}{2(N+1)} \frac{\partial}{\partial \mu} U_{\mu,a}|_{\mu=1,a=0} = \frac{1 - r^{2(N+1)}}{1 + r^{2(N+1)}}; \\ \phi_+ &= -\frac{1}{4} \frac{\partial}{\partial x} U_{\mu,a}|_{\mu=1,a=0} = \frac{r^{N+1}\cos((N+1)\theta)}{1 + r^{2(N+1)}}, \ (x = \mathcal{R}e\, a); \\ \phi_- &= -\frac{1}{4} \frac{\partial}{\partial y} U_{\mu,a}|_{\mu=1,a=0} = \frac{r^{N+1}\sin((N+1)\theta)}{1 + r^{2(N+1)}}, \ (y = \mathcal{I}m\, a). \end{aligned} \tag{3.4.24}$$

More interestingly, the following holds.

Proposition 3.4.19 *For $\alpha \in (0,1)$, the operator L in (3.4.22) satisfies:*
(a) $Ker L = span\{\phi_0, \phi_+, \phi_-\} \subset Y_\alpha$;
(b) $Im\, L = \{f \in X_\alpha : \int_{\mathbb{R}^2} f\phi_\pm = 0\}$.

To derive Proposition 3.4.19, we start by describing the behavior of the operator L over radial functions. Hence, denote by $L^r : Y_\alpha^r \to X_\alpha^r$ the operator

$$L^r\phi = \frac{d^2}{dr^2}\phi + \frac{1}{r}\frac{d}{dr}\phi + \rho\phi,\ \phi \in Y_\alpha^r, \tag{3.4.25}$$

where Y_α^r and X_α^r denote the subspaces in Y_α and X_α respectively restricted to radial functions.

Lemma 3.4.20 *Let $\alpha \in (0,1)$ and $n \in \mathbb{Z}^+$. Then:*
(a) $\phi \in Y_\alpha^r$ satisfies $L^r\phi = 0$ if and only if $\phi \in span\{\phi_0\}$.
(b) $L^r : Y_\alpha^r \to X_\alpha^r$ is onto. More precisely, for $f \in X_\alpha^r$, let

$$\begin{aligned} w(r) = &\left(\phi_0(r)\log r + \frac{2}{N+1}\frac{1}{\left(1+r^{2(N+1)}\right)}\right)\int_0^r \phi_0(t)f(t)t\,dt \\ &- \phi_0(r)\int_0^r\left(\phi_0(t)\log t + \frac{2}{N+1}\frac{1}{\left(1+t^{2(N+1)}\right)}\right)f(t)t\,dt. \end{aligned} \tag{3.4.26}$$

Then $w \in Y_\alpha^r$ and satisfies: $L^r w = f$.

Observe that $w(r)$ and $\dot{w}(r)$ extend with continuity at $r = 0$ where we find: $w(0) = 0 = \dot{w}(0)$. Furthermore, setting

$$c_f = \int_0^{+\infty} \phi_0(t)f(t)t\,dt, \tag{3.4.27}$$

we have:

Corollary 3.4.21 *The function w in (3.4.26) admits the following asymptotic behavior:*

$$w(r) = -c_f\log r + O(1),\ \text{as } r \to +\infty \tag{3.4.28}$$

$$\dot{w}(r) = -\frac{c_f}{r} + O(1),\ \text{as } r \to +\infty. \tag{3.4.29}$$

In particular, by taking $f(r) = \frac{1}{\lambda}\rho^2$ in (3.4.26) we obtain

$$\begin{aligned} w_0(r) = &\frac{1}{\lambda\left(1+r^{2(N+1)}\right)}\left[\left((1-r^{2(N+1)})\log r + \frac{2}{N+1}\right)\int_0^r \phi_0(t)t\rho^2(t)\,dt\right. \\ &\left. - \left(1-r^{2(N+1)}\right)\int_0^r\left(\phi_0(t)\log t + \frac{2}{(N+1)\left(1+t^{2(N+1)}\right)}\right)\rho^2(t)t\,dt\right]. \end{aligned} \tag{3.4.30}$$

The function in (3.4.30) defines a solution for problem (3.4.7) in Y_α^r, such that:

$$w_0(r) = -\frac{c_0}{\lambda}\log r + O(1), \qquad \text{as } r \to +\infty; \tag{3.4.31}$$

$$\dot{w}_0(r) = -\frac{1}{\lambda}\frac{c_0}{r} + O(1), \qquad \text{as } r \to +\infty, \tag{3.4.32}$$

with

$$c_0 = 16(N+1)^3 \int_0^{+\infty} \frac{s^{2N}}{(1+s^{N+1})^4}\, ds > 0. \tag{3.4.33}$$

Indeed, according to Corollary 3.4.21, we see that

$$\begin{aligned}
c_0 &= \int_0^{+\infty} \phi_0(t)\rho^2(t)t\, dt \\
&= \left(8(N+1)^2\right)^2 \int_0^{+\infty} \frac{\left(1-t^{2(N+1)}\right)}{\left(1+t^{2(N+1)}\right)^5} t^{4N+1}\, dt \\
&= 32(N+1)^4 \int_0^{+\infty} \frac{1-s^{N+1}}{(1+s^{N+1})^5} s^{2N}\, ds \\
&= 32(N+1)^4 \left(\int_0^{+\infty} \frac{s^{2N}}{(1+s^{N+1})^4} - 2\int_0^{+\infty} \frac{s^N}{(1+s^{N+1})^5} s^{2N+1} \right) \\
&= 32(N+1)^4 \left(\int_0^{+\infty} \frac{s^{2N}}{(1+s^{N+1})^4} + \frac{1}{2(N+1)} \int_0^{+\infty} \frac{d}{ds}\left(\frac{1}{(1+s^{N+1})^4}\right) s^{2N+1}\, ds \right) \\
&= 32(N+1)^4 \left(\int_0^{+\infty} \frac{s^{2N}}{(1+s^{N+1})^4}\, ds - \frac{2N+1}{2(N+1)} \int_0^{+\infty} \frac{s^{2N}}{(1+s^{N+1})^4} \right) \\
&= 16(N+1)^3 \int_0^{+\infty} \frac{s^{2N}}{(1+s^{N+1})^4}\, ds,
\end{aligned}$$

and (3.4.33) is established.

From now on we shall substitute such a solution w_0 into the definition of the operator P given in (3.4.10).

Proof of Lemma 3.4.20. To obtain (a), notice that if $\phi \in Y_\alpha^r$ satisfies

$$\ddot{\phi} + \frac{1}{r}\dot{\phi} + \rho\phi = 0,$$

then by standard (elliptic) regularity theory, we know that ϕ extends with continuity at $r = 0$. Consequently $r\dot{\phi} \in C^1[0, +\infty)$, and we obtain

$$\phi \in C^2(0, +\infty) \cap C^1[0, +\infty), \text{ and } \dot{\phi}(0) = 0.$$

Now write

$$\phi(r) = \phi_0(r)\psi(r),$$

so that $\psi \in C^2((0,1) \cup (1,+\infty)) \cap C^1([0,1) \cup (1,+\infty))$ satisfies:

$$\begin{cases} \ddot{\psi} + \dot{\psi}\left(\frac{1}{r} + 2\frac{\dot{\phi}_0}{\phi_0}\right) = 0 \\ \dot{\psi}(0) = 0. \end{cases} \tag{3.4.34}$$

Consequently $\frac{d}{dr}\left(r\phi_0^2(r)\dot{\psi}(r)\right) = 0$, that gives

$$\psi(r) = C \text{ for } 0 \le r < 1;$$

while for $r > 1$, we find

$$\dot{\psi}(r) = \frac{A}{r\phi_0^2(r)}, \text{ for a suitable constant } A \in \mathbb{R}. \tag{3.4.35}$$

That is

$$\psi(r) = A\left(\log r + \frac{2}{(N+1)\left(1 - r^{2(N+1)}\right)}\right) + B, \text{ for } r > 1 \tag{3.4.36}$$

for suitable constants $A, B, C \in \mathbb{R}$.

In other words:

$$\phi(r) = \begin{cases} C\phi_0(r), & 0 \le r < 1 \\ A\phi_0(r)\log r + \frac{2A}{(N+1)\left(1+r^{2(N+1)}\right)} + B\phi_0(r), & r > 1. \end{cases}$$

On the other hand, the continuity of ϕ at $r = 1$ requires that $A = 0$, while the continuity of $\dot{\phi}$ at $r = 1$ implies that $B = C$, and so $\phi \in \text{span}\{\phi_0\}$ as claimed.

To demonstrate that the formula (3.4.26) gives a solution w in Y_α^r for the non-homogeneous equation $L^r w = f$, we proceed as above and set

$$w = \phi_0 \psi.$$

Therefore $L^r w = f$, if and only if,

$$\frac{d}{dr}\left(r\phi_0^2(r)\dot{\psi}\right) = \phi_0(r) f(r) r.$$

Recalling (3.4.35) and (3.4.36), we may integrate the equation above for $0 < r < 1$ and using integration by parts to obtain a particular solution of the form:

$$\begin{aligned} \psi(r) &= \int_0^r \frac{1}{s\phi_0^2(s)}\left(\int_0^s \phi_0(t) f(t) t\, dt\right) ds \\ &= \left(\log r + \frac{2}{(N+1)\left(1 - r^{2(N+1)}\right)}\right)\int_0^r \phi_0(t) f(t) t\, dt \\ &\quad - \int_0^r \left(\phi_0(t)\log t + \frac{2}{(N+1)\left(1 + t^{2(N+1)}\right)}\right) f(t) t\, dt, \end{aligned}$$

where the integrals above are well-defined since $f \in X_\alpha^r$. Consequently,

$$w(r) = \phi_0(r)\psi(r) = \left(\phi_0(r)\log r + \frac{2}{(N+1)\left(1+r^{2(N+1)}\right)}\right)\int_0^r \phi_0(t) f(t)t\,dt$$
$$-\phi_0(r)\int_0^r \left(\phi_0(t)\log t + \frac{2}{(N+1)\left(1+t^{2(N+1)}\right)}\right) f(t)t\,dt.$$

Here $w(r)$ extends with continuity past the point $r = 1$ to define a solution for the equation $L^r w = f$, for every $r > 0$. And since $w(r)$ admits logarithmic growth as $r \to +\infty$ and $f \in X_\alpha$, we can ensure that it belongs to Y_α. □

Proof of Proposition 3.4.19. We start to establish (a). Letting $v \in Y_\alpha$ such that $Lv = 0$, then by standard elliptic regularity theory we see that $v \in C^2(\mathbb{R}^2)$. We write v according to its Fourier decomposition

$$v(z) = \sum_{k\in\mathbb{Z}} v_k(r)e^{ik\theta},\ z = re^{i\theta}, \tag{3.4.37}$$

with complex valued functions $v_k = v_k(r)$ such that $v_{-k} = \overline{v}_k$ and whose real and imaginary part satisfy

$$L^r(\phi) - \frac{k^2}{r^2}\phi = 0. \tag{3.4.38}$$

If $k = 0$, then for the real valued radial function $v_0(r) \in Y_\alpha^r$, we can use Lemma 3.4.20 to see that actually $v_0(r) \in \text{span}\{\phi_0\}$. For $k \in \mathbb{N}$, we are going to determine a fundamental set of solutions to (3.4.38) by using a family of solutions for the (singular) Liouville equations. More precisely, for $a \in \mathbb{C}$ and $k \in \mathbb{N}$, let

$$\psi_{a,k}(z) = \log \frac{8|(N+1)z^N + (k+N+1)az^{N+k}|^2}{\left(1+|z^{N+1}+az^{N+k+1}|^2\right)^2},$$

so that $\psi_{a=0,k} = \log\rho$. According to (2.2.3), $\psi_{a,k}$ satisfies

$$-\Delta\psi_{a,k} = e^{\psi_{a,k}} - 4\pi N\delta_{z=0} - 4\pi\sum_{j=1}^{k}\delta_{z_j^a},\ \text{in } \mathbb{R}^2,$$

where z_j^a, $j = 1,\ldots,k$ correspond to the k-distinct *non-zero* roots of the polynomial $(N+1)z^N + (k+N+1)az^{N+k}$. Notice in particular that $|z_j^a| \to +\infty$ as $a \to 0$, $\forall j = 1,\ldots,k$. Therefore, for each test function $\varphi \in C_0^\infty(\mathbb{R}^2)$, and $|a|$ sufficiently small, we have:

$$-\Delta\psi_{a,k}\varphi = e^{\psi_{a,k}}\varphi - 4\pi N\varphi(0).$$

So we can differentiate this expression at $a = 0$ with respect to $x = \mathcal{R}e\ a$ and $y = \mathcal{I}m\ a$, and obtain that

$$\varphi_{1,k} = \left.\frac{\partial\psi_{a,k}}{\partial x}\right|_{a=0} = \frac{2}{N+1}\phi_k(r)\cos k\theta;\ \varphi_{2,k} = \left.\frac{\partial\psi}{\partial y}\right|_{a=0} = \frac{2}{N+1}\phi_k(r)\sin k\theta,$$

with

$$\phi_k(r) = \frac{k+N+1+(k-N-1)r^{2(N+1)}}{1+r^{2(N+1)}} r^k, \tag{3.4.39}$$

satisfy

$$L\varphi_{1,k} = 0 = L\varphi_{2,k}, \; \forall k \in \mathbb{N}.$$

As a consequence, we obtain that $\phi_k(r)$ satisfies (3.4.38). In addition, by replacing k with $-k$ in (3.4.39) we still obtain a solution $\tilde{\phi}_k$ for (3.4.38) and we check that $\tilde{\phi}_k(r) = \phi_k\left(\frac{1}{r}\right)$. Thus, $\phi_k(r)$ and $\tilde{\phi}_k(r)$ define a fundamental set of solutions for (3.4.38). On the other hand, the real and imaginary part of v_k cannot include the component $\tilde{\phi}_k(r)$, which admits a $\frac{1}{r^k}$ singularity at the origin, $\forall k \in \mathbb{N}$. Furthermore, for $\alpha \in (0,1)$ and $k \neq N+1$, the function $\phi_k \notin Y_\alpha^r$ since it behaves as r^k, as $r \to +\infty$. So we conclude that $v_k = 0$, $\forall k \neq N+1$. Finally, for $k = N+1$ we see that $\mathcal{R}e\,(v_{k=N+1})$ and $\mathcal{I}m(v_{k=N+1})$ belong to $\text{span}\{\phi_{N+1}(r)\} = \text{span}\left\{\frac{r^{N+1}}{1+r^{2(N+1)}}\right\}$. We conclude that $v \in \text{span}\{\phi_0, \phi_+, \phi_-\}$ as claimed.

To establish (b), we start by observing the following fact:

Claim: The range of L is closed in X_α.
Let $\varphi_n \in (KerL)^\perp \subset Y_\alpha$ be such that $L\varphi_n = f_n \in X_\alpha$ and $f_n \to f$ in X_α. We claim that φ_n is uniformly bounded in Y_α. This is equivalent to saying that $\int_{\mathbb{R}^2} \frac{\varphi_n^2}{(1+|z|^{\alpha+2})} \leq C$, $\forall n \in \mathbb{N}$, for suitable $C > 0$. Arguing by contradiction, assume that (along a subsequence)

$$c_n = \left(\int_{\mathbb{R}^2} \frac{\varphi_n^2}{(1+|z|^{\alpha+2})}\right)^{\frac{1}{2}} \to +\infty, \text{ and set } \phi_n = \frac{\varphi_n}{c_n}.$$

So that,

$$\int_{\mathbb{R}^2} \frac{\phi_n^2}{(1+|z|^{\alpha+2})} = 1, \; \phi_n \in (KerL)^\perp \subset Y_\alpha, \text{ and } L\phi_n \to 0 \text{ in } X_\alpha. \tag{3.4.40}$$

As a consequence of (3.4.40), we see that ϕ_n is uniformly bounded in Y_α. Hence, we find $\phi \in Y_\alpha$ such that along a subsequence, we have

$$\phi_n \to \phi \text{ weakly in } Y_\alpha,$$

and we can assume further that the convergence above also holds in $L^2_{\text{loc}}(\mathbb{R}^2)$.

Since

$$|\phi_n(z) - \phi(z)| \leq c(||\phi_n||_{Y_\alpha} + ||\phi||_{Y_\alpha}) \log(1+|z|) \leq C \log(1+|z|),$$

we see that

$$\int_{\mathbb{R}^2} \frac{(\phi_n - \phi)^2}{(1+|z|^{1+\alpha})^2} \leq ||\phi_n - \phi||^2_{L^2(B_R)} + C\int_{|z|\geq R} \frac{\log(1+|z|)}{(1+|z|^{1+\alpha})^2} = ||\phi_n - \phi||^2_{L^2(B_R)} + o(1),$$

as $R \to +\infty$. Thus, $\int_{\mathbb{R}^2} \frac{(\phi_n - \phi)^2}{(1+|z|^{1+\alpha})^2} \to 0$ as $n \to \infty$, and we conclude that

$$\int_{\mathbb{R}^2} \frac{|\phi|^2}{(1+|z|^{1+\alpha})^2} = 1. \tag{3.4.41}$$

Similarly we see that, $\int_{\mathbb{R}^2} \rho\phi_n \to \int_{\mathbb{R}^2} \rho\phi$ as $n \to \infty$. Hence, $L\phi = 0$ and $\phi \in (KerL)^\perp$, that is, $\phi = 0$ in contradiction to (3.4.41). In conclusion, the sequence $\varphi_n \in (KerL)^\perp$ is uniformly bounded in Y_α, and we can argue exactly as above to find $\varphi \in Y_\alpha$ such that $\varphi_n \to \varphi$ weakly in Y_α with $L\varphi = f$, and the Claim is proved.

Consequently, we may decompose

$$X_\alpha = ImL \oplus (ImL)^\perp,$$

according to the scalar product in X_α.

Thus, for $\xi \in (ImL)^\perp$ we have

$$0 = (Lu, \xi)_{X_\alpha} = (Lu, (1+|z|)^{2+\alpha}\xi), \ \forall u \in Y_\alpha.$$

The density of C_0^∞ in Y_α implies that

$$\psi = (1+|z|)^{2+\alpha}\xi \in Y_\alpha \text{ and } L\psi = 0.$$

Therefore by part (a), we may write: $\psi = a_0\phi_0 + a_+\phi_+ + a_-\phi_-$ for suitable constants a_0, a_+ and a_-. Furthermore, by Lemma 3.4.20, we know that the radial function $f(r) = \phi_0(r)\frac{1}{(1+r^2)^2} \in ImL$ and satisfies $(f, \phi_\pm)_{L^2} = 0$. Consequently,

$$0 = (f, \xi)_{X_\alpha} = (f, \psi)_{L^2} = a_0\left(2\pi \int_0^{+\infty} \left(\frac{\phi_0(r)}{1+r^2}\right)^2 r\,dr\right),$$

that is, $a_0 = 0$, and part (b) of our statement follows. □

At this point we can complete our perturbation analysis. By virtue of (3.4.30) and (3.4.33), we see that

$$P : Y_\alpha \times \mathbb{C} \times \mathbb{R} \longrightarrow X_\alpha$$

in (3.4.10) is a well-defined smooth operator that can be extended with continuity at $\varepsilon = 0$, where we have $P(0,0,0) = 0$. Moreover, the linearized operator

$$A : \frac{\partial P}{\partial(u_1, a)}(0,0,0) : Y_\alpha \times \mathbb{C} \longrightarrow X_\alpha \tag{3.4.42}$$

takes the form

$$A(\varphi, b) = L\varphi + M(b), \tag{3.4.43}$$

with $\varphi \in Y_\alpha$, $b = b_1 + ib_2 \in \mathbb{C}$ and

$$M(b) = -4\left(\rho w_0 - \frac{2}{\lambda}\rho^2\right)\phi_+ b_1 - 4\left(\rho w_0 - \frac{2}{\lambda}\rho^2\right)\phi_- b_2. \tag{3.4.44}$$

So on the basis of Proposition 3.4.19, we expect also to characterize $KerA \subset Y_\alpha$ and $ImA \subset X_\alpha$. To this purpose, we observe:

Lemma 3.4.22

$$\int_{\mathbb{R}^2}\left(\rho w_0-\frac{2}{\lambda}\rho^2\right)\phi_\pm^2=\pi\int_0^{+\infty}\left(\rho(r)w_0(r)-\frac{2}{\lambda}\rho^2(r)\right)\frac{r^{2(N+1)}}{\left(1+r^{2(N+1)}\right)^2}r\,dr<0.$$

Proof. We start by observing that

$$L^r\left(\frac{1}{16\left(1+r^{2(N+1)}\right)^2}\right)=\frac{(N+1)^2r^{4N+2}}{\left(1+r^{2(N+1)}\right)^4}.$$

Therefore, in view of the decay estimates (3.4.31) and (3.4.32) of w_0, we can use integration by parts to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(\rho w_0-\frac{2}{\lambda}\rho^2\right)\phi_\pm^2\\
&=\pi\int_0^{+\infty}\left(\frac{8(N+1)^2r^{4N+2}}{\left(1+r^{2(N+1)}\right)^4}w_0(r)-\frac{2}{\lambda}\rho^2\frac{r^{2(N+1)}}{\left(1+r^{2(N+1)}\right)^2}\right)rdr\\
&=\pi\int_0^{+\infty}\left(\frac{1}{2}L^r\left(\frac{1}{\left(1+r^{2(N+1)}\right)^2}\right)w_0(r)-\frac{2}{\lambda}\rho^2\frac{r^{2(N+1)}}{\left(1+r^{2(N+1)}\right)^2}\right)rdr\\
&=\int_0^{+\infty}\left(\frac{1}{2}L^r w_0\frac{1}{\left(1+r^{2(N+1)}\right)^2}-\frac{2}{\lambda}\rho^2\frac{r^{2(N+1)}}{\left(1+r^{2(N+1)}\right)^2}\right)rdr\\
&=\frac{1}{\lambda}\int_0^{+\infty}\rho^2(r)\left(\frac{1}{2\left(1+r^{2(N+1)}\right)^2}-\frac{2r^{2(N+1)}}{\left(1+r^{2(N+1)}\right)^2}\right)rdr,
\end{aligned}$$

where, in the last identity, we have used the fact that w_0 satisfies (3.4.7). To estimate the last integral above, we use the change of variable $t=r^2$ to find

$$\begin{aligned}
&\int_0^{+\infty}\rho^2(r)\frac{1-4r^{2(N+1)}}{2\left(1+r^{2(N+1)}\right)^2}r\,dr\\
&=16(N+1)^4\int_0^{+\infty}\frac{1-4t^{N+1}}{(1+t^{N+1})^6}t^{2N}\,dt\\
&=16(N+1)^4\left(\int_0^{+\infty}\frac{t^{2N}}{(1+t^{N+1})^5}-5\int_0^{+\infty}\frac{t^{3N+1}}{(1+t^{N+1})^6}\,dt\right)\\
&=16(N+1)^4\left(\int_0^{+\infty}\frac{t^{2N}}{(1+t^{N+1})^5}+\frac{1}{N+1}\int_0^{+\infty}t^{2N+1}\frac{d}{dt}\left(\frac{1}{(1+t^{N+1})^5}\right)dt\right)\\
&=16(N+1)^4\left(\int_0^{+\infty}\frac{t^{2N}}{(1+t^{N+1})^5}-\frac{2N+1}{N+1}\int_0^{+\infty}\frac{t^{2N}}{(1+t^{N+1})^5}\,dt\right)\\
&=-16(N+1)^3N\int_0^{+\infty}\frac{t^{2N}}{(1+t^{N+1})^5}\,dt<0,
\end{aligned}$$

and the desired conclusion follows. □

We are now ready to conlcude:

Proposition 3.4.23 *For* $\alpha \in (0,1)$*, the operator* $A : Y_\alpha \to X_\alpha$ *in (3.4.42) and (3.4.43) is* onto *and* $KerA = KerL = span\{\phi_0, \phi_+, \phi_-\}$.

Proof. Let $f \in X_\alpha$, we need to find $\varphi \in Y_\alpha$ and $b = b_1 + ib_2 \in \mathbb{C}$ such that

$$A(\varphi, b) = L\varphi + M(b) = f, \text{ with } M(b) \text{ in (3.4.44).} \tag{3.4.45}$$

To this purpose, multiply (3.4.45) by ϕ_+ and integrate over $\mathbb{R}^2$ to find

$$\begin{aligned}\int_{\mathbb{R}^2} f\phi_+ &= \int_{\mathbb{R}^2} L\varphi\phi_+ - 4b_1 \int_{\mathbb{R}^2} \left(\rho w_0 - \frac{2}{\lambda}\rho^2\right)\phi_+^2 - 4b_2 \int_{\mathbb{R}^2} \left(\rho w_0 - \frac{2}{\lambda}\rho^2\right)\phi_+\phi_- \\ &= -4\pi b_1 \int_0^{+\infty} \left(\rho(r)w_0(r) - \frac{2}{\lambda}\rho^2(r)\right) \frac{r^{2(N+1)}}{(1+r^{2(N+1)})^2}\, dr,\end{aligned}$$

as follows by Proposition 3.4.19 (b), and the well-known orthogonality properties of trigonometric functions.

On the basis of Lemma 3.4.22, we may solve for b_1 and derive:

$$b_1 = -\frac{1}{4\pi}\int_{\mathbb{R}^2} f\phi_+ \left(\int_0^{+\infty} \left(\rho(r)w_0(r) - \frac{2}{\lambda}\rho^2(r)\right) \frac{r^{2(N+1)}}{(1+r^{2(N+1)})^2}\, dr\right)^{-1}. \tag{3.4.46}$$

Analogously, multiplying (3.4.45) by ϕ_- and integrating over $\mathbb{R}^2$ we derive:

$$b_2 = -\frac{1}{4\pi}\int_{\mathbb{R}^2} f\phi_- \left(\int_0^{+\infty} \left(\rho(r)w_0(r) - \frac{2}{\lambda}\rho^2(r)\right) \frac{r^{2(N+1)}}{\left(1+r^{2(N+1)}\right)^2}\, dr\right)^{-1}. \tag{3.4.47}$$

Set $g = f - M(b)$, with $M(b)$ in (3.4.44) and b_1 and b_2 specified, respectively, in (3.4.46) and (3.4.47). We easily check that, $\int_{\mathbb{R}^2} g\phi_\pm = 0$. Our problem is now reduced to finding $\varphi \in X_\alpha : L\varphi = g$, and this we have by Proposition 3.4.19 (b). Thus, we have deduced that $ImA = X_\alpha$. To characterize $KerA$, just take $f = 0$ in the argument above to find that $b_1 = 0 = b_2$, and so $A\varphi = 0$, if and only if, $L\varphi = 0$; that is $KerA = KerL = \text{span}\{\phi_0, \phi_+, \phi_-\}$. □

Setting

$$U_\alpha = \text{ span}\{\phi_0, \phi_+, \phi_-\}^\perp,$$

we obtain the following existence result for (3.4.4):

Proposition 3.4.24 *For every* $\lambda > 0$ *and* $\alpha \in (0,1)$*, there exist* $\varepsilon_0 > 0$ *sufficiently small and smooth functions*

$$a_\varepsilon : (-\varepsilon_0, \varepsilon_0) \to \mathbb{C},\; u_{1,\varepsilon} : (-\varepsilon_0, \varepsilon_0) \to U_\alpha,$$

with $a_{\varepsilon=0} = 0$ *and* $u_{1,\varepsilon=0} = 0$*, such that the function*

$$u_\varepsilon(z) = u^0_{\varepsilon,a_\varepsilon}(\varepsilon z) + \log\varepsilon^2 + \varepsilon^2 w_0(\varepsilon|z|) + \varepsilon^2 u_{1,\varepsilon}(\varepsilon z) \tag{3.4.48}$$

defines a solution for (3.4.4), with w_0 defined in (3.4.30) and satisfying (3.4.31)-(3.4.33). Furthermore as $\varepsilon \to 0$, the following estimates hold:

$$|u_{1,\varepsilon}(z)| \le o(1)\log(1+|z|),\ |\nabla u_{1,\varepsilon}(z)| = \tfrac{o(1)}{1+|z|},\ \forall z \in \mathbb{R}^2;$$
$$\lambda \int_{\mathbb{R}^2} e^{u_\varepsilon}(1-e^{u_\varepsilon}) = 8\pi(N+1) + o(1). \tag{3.4.49}$$

Proof. A straightforword application of the Implicit Function Theorem (cf. [Nir]) yields to u_ε in (3.4.48). So it remains to check the validity of (3.4.49). To this purpose, from (3.4.12), we have

$$|u_{1,\varepsilon}(z)| \le C_1 ||u_{1,\varepsilon}||_{Y_\alpha} \log(1+|z|),\ \forall z \in \mathbb{R}^2,$$

and $||u_{1,\varepsilon}||_{Y_\alpha} \to 0$, as $\varepsilon \to 0$. Furthermore, from (3.4.10) we see that,

$$-\Delta u_{1,\varepsilon} = f_{1,\varepsilon},$$

with $f_{1,\varepsilon} \to 0$ in X_α. We can check also that $(1+|z|^2)|f_{1,\varepsilon}| \le c_\lambda \varepsilon$ in $\mathbb{R}^2$, with a suitable constant c_λ depending on λ but independent of ε. Since $u_{1,\varepsilon} \in Y_\alpha$, we can write

$$\nabla u_{1,\varepsilon}(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{y-z}{|y-z|^2} f_{1,\varepsilon}(y)\,dy,$$

and for $|z| \ge 2$ derive the following estimate:

$$\begin{aligned}
|z||\nabla u_{1,\varepsilon}(z)| &\le \int_{\{|y-z|\le \frac{|z|}{2}\}} \frac{|z||f_{1,\varepsilon}(y)|}{|y-z|} + \int_{\{|y-z|\ge \frac{|z|}{2}\}} \frac{|z||f_{1,\varepsilon}(y)|}{|y-z|} \\
&\le 2\int_{\{|y-z|\le \frac{|z|}{2}\}} \frac{|y||f_{1,\varepsilon}(y)|}{|y-z|} \\
&\quad +2\int_{\{|y-z|\le \frac{|z|}{2}\}} |f_{1,\varepsilon}(y)| + \pi|z|\left(\max_{\{|y|\ge \frac{|z|}{2}\}} |y||f_{1,\varepsilon}(y)|\right) + 2||f_{1,\varepsilon}||_{L^1(\mathbb{R}^2)} \\
&\le c_\lambda \varepsilon + 2||f_{1,\varepsilon}||_{L^1(\mathbb{R}^2)} \to 0, \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

The first estimate in (3.4.49) then follows.

Finally, notice that

$$\begin{aligned}
\lambda \int_{\mathbb{R}^2} e^{u_\varepsilon(z)}(1-e^{u_\varepsilon(z)}) &= \lambda \int_{\mathbb{R}^2} e^{u_\varepsilon(\frac{z}{\varepsilon})}(1-e^{u_\varepsilon(\frac{z}{\varepsilon})})\frac{1}{\varepsilon^2} \\
&= \lambda \int_{\mathbb{R}^2} e^{u_{\varepsilon,a_\varepsilon}(z)+\varepsilon^2(w_0+u_{1,\varepsilon})} - \lambda\varepsilon^2 \int_{\mathbb{R}^2} e^{2u^0_{\varepsilon,a_\varepsilon}(z)+2\varepsilon^2(w_0+u_{1,\varepsilon})}.
\end{aligned}$$

Therefore, we can use the available estimates and (by dominated converge) pass to the limit into the integral sign and conclude:

$$\begin{aligned}
\lim_{\varepsilon\to 0} \lambda \int_{\mathbb{R}^2} e^{u_\varepsilon(z)}(1-e^{u_\varepsilon(z)}) = \lambda \int_{\mathbb{R}^2} e^{u^0_{\varepsilon=0,a=0}} &= 16\pi(N+1)\int_0^{+\infty} \frac{r^{2N+1}}{\left(1+r^{2(N+1)}\right)^2}\,dr \\
&= 8\pi(N+1)\int_0^{+\infty} \frac{dt}{(1+t)^2} = 8\pi(N+1).
\end{aligned}$$

□

By virtue of Proposition 3.4.24, we conclude the following existence result concerning non-topological Chern–Simons vortices:

Theorem 3.4.25 *For a given set of (vortex) points* $Z = \{z_1, \dots, z_N\}$ *(repeated according to their multiplicity) and* $k > 0$*, there exist* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0)$*, we have a planar vortex configuration* $(\mathcal{A}^\varepsilon, \phi^\varepsilon)_\pm$ *solution to the self-dual equations (1.2.45) in* $\mathbb{R}^2$ *(with the* $\pm$ *sign choosen accordingly) such that*
i) $\phi^\varepsilon_\pm$ *vanishes exactly in the set Z, and if* $n_j \in \mathbb{N}$ *is the multiplicity of* z_j*, then*

$$\phi^\varepsilon_+(z) \text{ and } \overline{\phi}^\varepsilon_-(z) = O((z - z_j)^{n_j}), \text{ as } z \to z_j, \ j = 1, \dots, N. \qquad (3.4.50)$$

ii) $\phi^\varepsilon_\pm$ *satisfies property (3.2.1), and* $\phi^\varepsilon_\pm(z) \to 0$ *as* $|z| \to +\infty$*. More precisely, there exist constants* $C_\varepsilon > 0$*,* $R_\varepsilon > 0$ *and* $\beta_\varepsilon \to 0^+$ *as* $\varepsilon \to 0$ *such that*

$$|F^\varepsilon_{12}| + |z|^2 |\nabla |\phi^\varepsilon_\pm||^2 + |\phi^\varepsilon_\pm|^2 \le C_\varepsilon |z|^{-2(N+2+\beta_\varepsilon)}, \ \forall |z| \ge R_\varepsilon. \qquad (3.4.51)$$

iii) Magnetic flux:

$$\Phi^\varepsilon_\pm = \int_{\mathbb{R}^2} (F^\varepsilon_{12})_\pm = \pm 4\pi (N+1) + o(1); \qquad (3.4.52)$$

Electric charge:

$$Q^\varepsilon_\pm = \int_{\mathbb{R}^2} (J^0_\varepsilon)_\pm = \pm 4\pi k(N+1) + o(1); \qquad (3.4.53)$$

Total energy:

$$E_\varepsilon = \int_{\mathbb{R}^2} \mathcal{E}_\pm = 4\pi \nu^2 (N+1) + o(1); \qquad (3.4.54)$$

as $\varepsilon \to 0$.

Observe that in comparison to Theorem 3.2.3, the construction above yields to a vortex configuration that verifies the "concentration" property (3.2.8) only when all the vortex points coincide. Indeed, to fix the ideas, let $z_1 = z_2 = \cdots = z_N = 0$, $\nu = 1$ and $\lambda = 1$, then by Lemma 3.4.20 we can argue as above to obtain a *radial* solution (about the origin) for the equation

$$-\Delta u = e^u (1 - e^u) - 4\pi N \delta_{z=0}, \ \text{ in } \mathbb{R}^2, \qquad (3.4.55)$$

in the form

$$u_\varepsilon(r) = \log\left(\frac{8(N+1)^2 \varepsilon^{2(N+1)} r^{2N}}{(1 + \varepsilon^{2(N+1)} r^{2(N+1)})^2} \right) + \varepsilon^2 w_0(\varepsilon r) + \varepsilon^2 u_{1,\varepsilon}(\varepsilon r), \qquad (3.4.56)$$

with w_0 defined in (3.4.30) and $u_{1,\varepsilon}$ satisfying the estimates in (3.4.49). Consequently, for $k > 0$, the function

$$u_{\varepsilon,k}(r) = u_\varepsilon \left(\frac{2}{k} r \right) \qquad (3.4.57)$$

defines a (radial) solution to the problem:

$$\begin{cases} -\Delta u = \frac{4}{k^2} e^u (1 - e^u) - 4\pi N \delta_{z=0} \text{ in } \mathbb{R}^2 \\ u(z) \to 0 \text{ as } |z| \to +\infty. \end{cases} \tag{3.4.58}$$

Thus, if we choose $\varepsilon = \varepsilon_k : \frac{\varepsilon_k}{k} \to 0$, as $k \to 0$, and set $u_k = u_{\varepsilon_k, k}$, then

$$\frac{4}{k^2} e^{u_k} \to 8\pi(N+1)\delta_{z=0}, \text{ weakly in the sense of measure in } \mathbb{R}^2, \text{ as } k \to 0^+;$$

and the corresponding non-topological Chern–Simons radial vortex configuration satisfies (3.2.8) with $z_1 = z_2 = \cdots = z_N = 0$.

We recall that a class of radial "non-topological" vortices was constructed for the first time by Spruck and Yang in [SY1] by using a "shooting" method for the corresponding O.D.E. problem. In particular, in [SY1] it was shown that (for $\nu = 1$) the energy $\mathcal{E}$ of radial non-topological solutions satisfies the lower bound $\mathcal{E} > 4\pi(N+1)$, which we can now assert to be sharp.

However to have non-topological vortices enhanced with property (3.2.8), (as it would be desirable for the physical applications) seems to require a bigger amount of energy, i.e., $\mathcal{E} > 8\pi N$. This fact was observed by Chan–Fu–Lin in [CFL], who introduced an alternative construction of non-topological vortices related to a new class of radial solutions for (3.4.55). To this purpose, we observe that the radial solution u_ε in (3.4.56) satisfies:

$$\int_0^{+\infty} e^{u_\varepsilon(r)}(1 - e^{u_\varepsilon(r)}) r\, dr = 4(N+1) + o(1) \text{ as } \varepsilon \to 0.$$

In [CFL] the authors show that, in fact, for any prescribed $\beta > 4(N+1)$, there exists a *unique* radial solution $u = u(r)$ for the equation (3.4.55) satisfying:

$$\int_0^{+\infty} e^{u(r)}(1 - e^{u(r)}) r\, dr = \beta$$

(see Theorem 2.1 in [CFL]).

Hence, Chan–Fu–Lin in [CFL] search for the solution of (3.1.4), whose profile (after suitable scaling and translation) around a vortex point looks like one such "new" radial solution.

However, to attain such "concentration" property, the authors need to require a specific location for the vortex points $\{z_1, \ldots, z_N\}$, which must be placed around a circle to form an equilateral polygon with $\mathbb{Z}_N$-symmetry. Thus, if z_0 is the center of the circle, then the angle between the segment $\overline{z_0 z_j}$ and $\overline{z_0 z_{j+1}}$ must be equal to $\frac{2\pi}{N}$, $\forall j = 1, \ldots, N$ and $z_{N+1} = z_1$. The solution constructed in this way keeps the same $\mathbb{Z}_N$-symmetry.

More precisely, under such assumptions on the location of the vortex points and the normalization $\nu = 1$, the following holds:

Theorem 3.4.26 *[CFL]: Given a number $\Phi > 8\pi N$, there exists $k_0 > 0$ such that, for every $k \in (0, k_0)$, we have a planar selfdual vortex configuration $(\mathcal{A}, \phi)_\pm$ solution to (1.2.45) in $\mathbb{R}^2$ (with $\pm$ sign chosen accordingly and $\nu = 1$) with the following properties:*
i) $\phi_\pm$ vanishes exactly at the $\mathbb{Z}_N$-symmetrically located (distinct) points $z_1, \ldots, z_N$ and

$$\phi_+(z),\ \overline{\phi}_-(z) = O(z - z_j) \text{ as } z \to z_j,\ j = 1, \ldots, N.$$

Furthermore $|\phi_\pm| < 1$ in $\mathbb{R}^2$ and $|\phi_\pm| \to 0$ as $|z| \to +\infty$.
ii) The following estimate holds:

$$|(F_{12})_\pm| + |z|^2 |\nabla |\phi_\pm||^2 + |\phi_\pm|^2 = O\left(|z|^{-2\left(\frac{\Phi}{2\pi} - N\right)}\right), \text{ as } |z| \to +\infty.$$

iii) The corresponding magnetic flux, electric charge and total energy are given respectively as follows:
Magnetic flux: $\int_{\mathbb{R}^2} (F_{12})_\pm = \pm\Phi$; Electric charge: $\int_{\mathbb{R}^2} (J^0)_\pm = \pm k\Phi$;
Total energy: $\int_{\mathbb{R}^2} \mathcal{E}_\pm = \Phi$.

In fact, for the local flux around z_j, $j = 1, \ldots, N$, we have

$$\int_{B_{\delta_0}(z_j)} (F_{12})_\pm = \pm \frac{1}{N}\Phi + o(1), \text{ as } k \to 0^+,$$

for sufficiently small $\delta_0 > 0$.
iv) As $k \to 0^+$, we have:
(a) $(F_{12})_\pm \to \pm \left(\frac{1}{N}\Phi\right) \sum_{j=1}^N \delta_{z_j}$, weakly in the sense of measure;
(b) $\log |\phi|^2$ is locally radially symmetric around z_j, $j = 1, \ldots, N$, and after suitable scaling,

$$\log |\phi|^2 \to \left(2 - \frac{\Phi}{2\pi N}\right) \sum_{j=1}^N \log |z - z_j|^2 + C,$$

for some $C \in \mathbb{R}$.

We refer to [CFL] for details.

3.5 Final remarks and open problems

In concluding our discussion about planar Chern–Simons vortices, we wish to return and emphasize the main problems which remain *open* for the planar 6th-order Chern–Simons model discussed above.

Firstly, concerning planar topological solutions, their unique and possibly smooth dependence on the vortex points remains to be clarified. This question can be formulated more precisely in terms of the elliptic problem

$$\begin{cases} -\Delta u = e^u(1-e^u) - 4\pi \sum_{j=1}^{N} \delta_{z_j} \quad \text{in } \mathbb{R}^2 \\ u(x) \to 0 \quad \text{as } |x| \to +\infty \end{cases} \tag{3.5.1}$$

as follows:

Open problem: For any assigned set of (vortex) points $\{z_1, \ldots, z_N\} \subset \mathbb{R}^2$ (not necessarily distinct), does problem (3.5.1) admit a *unique* solution? Can it be smoothly parametrized by the given points $\{z_1, \ldots, z_N\}$?

We know the answer to be affirmative for the case $z_1 = \cdots = z_N$, namely, when all the vortex points are superimposed with multiplicity $N \in \mathbb{N}$. In fact, in this case one can provide uniqueness for problem (3.3.1) for any $N \in [0, +\infty)$, and in particular, for $N = 0$, where Dirac measures are not included in (3.5.1).

Such uniqueness property persists under small "perturbations," in the sense that, uniqueness continues to hold for (3.5.1) when all vortex points are located sufficiently close to the origin ("perturbation" from the single vortex situation) or to infinity ("perturbation" from the zero vortex situation) (see [Cho]).

Concerning planar non-topological vortices, it is important to understand if the limit, $k \to 0$, produces a "concentration" effect on the vortex configuration, as it always occurs for topological vortices.

In other words, we are now interested in exploring the possibility of constructing a solution u_λ for the problem

$$\begin{cases} -\Delta u = \lambda e^u(1-e^u) - 4\pi \sum_{j=1}^{N} \delta_{z_j} \\ u(x) \to -\infty \quad \text{as } |x| \to +\infty \end{cases}$$

such that

$$\lambda e^{u_\lambda}(1-e^{u_\lambda}) \to \sum_{j=1}^{N} \beta_j \delta_{z_j}, \quad \text{as } \lambda \to +\infty, \tag{3.5.2}$$

weakly in the sense of measure (possibly along a sequence $\lambda = \lambda_n \to +\infty$), for suitable $\beta_j > 0$, $j = 1, \ldots, N$.

Again, a partial answer can be furnished when all the vortex points coincide, i.e., $z_1 = \cdots = z_N$ and $\beta = 8\pi(N+1)$ (see (3.4.56) and (3.4.57)), or when they are all distinct and placed to form a $\mathbb{Z}_N-$symmetric polygon in $\mathbb{R}^2$. In this case, (3.5.2) is satisfied for *any* choice of $\beta = \beta_1 = \cdots = \beta_N > 16\pi$.

Finally, all our analysis has not yet clarified whether a propriety analogous to that stated in Theorem 3.2.2 for the Maxwell–Higgs vortices remains valid for Chern–Simons vortices.

Namely,

Open problem: Does any finite energy (static) solution of the Chern–Simons field equations (1.2.33) and (1.2.34) in $\mathbb{R}^2$ reduce to a solution of the selfdual equation (1.2.45)?

An affirmative answer to this question would show that no mixed vortex-antivortex configurations occur for the selfdual Chern–Simons model discussed above, in a manner similar to what happens for the Maxwell–Higgs model.

It would be very interesting to provide an answer to such a question, were it limited to Chern–Simons solutions subject to topological boundary condition at infinity, or periodic ones (as discussed in the following chapter).

Concerning other Chern–Simons models, we mention that the existence of selfdual Maxwell–Chern–Simons–Higgs vortex configurations $(\mathcal{A}, \phi, N)$ satisfying (1.2.63) under *topological*-type boundary conditions has been established by Chae–Kim in [ChK1]. Since the topological boundary conditions require a non-vanishing property for the magnitude of the Higgs field at infinity, we see that (by taking into account (1.2.59), (1.2.61), and (1.2.62)) for the Maxwell–Chern–Simons–Higgs model, this amounts to satisfy

$$|\phi|^2 \to \nu^2 \text{ and } N \to \frac{\nu^2}{k}, \text{ as } |z| \to +\infty; \tag{3.5.3}$$

while *non-topological* boundary conditions are expressed as

$$|\phi|^2 \to 0 \text{ and } N \to 0, \text{ as } |z| \to +\infty. \tag{3.5.4}$$

In [ChK1], the authors succeded in showing that under (3.5.3) the corresponding elliptic problem (2.1.14) can be formulated variationally over the space $H^2(\mathbb{R}^2)$. Thus, in analogy to the 6th-order Chern–Simons model, they obtain a topological solution by a minimization procedure. For this solution, the rate of convergence in (3.5.3) is exponentially fast. In addition, the authors also use an iteration scheme to obtain an alternative existence result yielding to a solution having the advantage to pass to the limit in strong norm in the abelian–Higgs limit (i.e., $\sigma \to 0$ and q fixed), and in weak norm in the Chern–Simons limit (i.e., $\sigma \to 0$ and $q \to +\infty$ and $k = \frac{\sigma}{q^2}$ fixed).

It is interesting to note that an analogous variational approach works equaly well to treat planar topological vortices for the non-abelian Chern–Simons model (1.3.99), (1.3.100), and (1.3.101). In this case, under the ansatz (1.3.116), (1.3.117), the finite energy condition imposes that the component ϕ^a of the Higgs field satisfies

$$\int_{\mathbb{R}^2} |\phi^a|^2(\nu^2 - |\phi^b|^2 K_{ba}) < +\infty, \tag{3.5.5}$$

$\forall a = 1, \ldots, r$, where r is the rank of the (semisimple) gauge group with associated Cartan matrix $K = (K_{ba})$.

Hence in this situation, *topological* vortices are require to satisfy

$$|\phi^a|^2 \to |\phi_0^a|^2 := \nu^2 \sum_{b=1}^{r} (K^{-1})_{ab}, \text{ as } |z| \to +\infty, \tag{3.5.6}$$

for $a = 1, \ldots, r$.

In [Y6], Y. Yang introduced a variational principle (in the Sobolev space $H^1(\mathbb{R}^2, \mathbb{R}^r)$ of vector-valued functions) corresponding to the elliptic system (2.1.21), subject to the boundary condition (3.5.6). Again, by a minimization principle, the author in [Y6] derives a topological non-abelian Chern–Simon vortex in $\mathbb{R}^2$, characterized by properties analogous to those established above for the topological vortex solution of the 6th-order Chern–Simons model (describing the case for rank $r = 1$). We refer to [Y6] or [Y1] for details.

Observe that the element $\phi_0 = \phi_0^a E_a$, with $|\phi_0^a|^2$ in (3.5.6), defines a zero for the (non-abelian) selfdual potential V in (1.3.101). Hence, it defines a vacuum state for the system, known as the *principal embedding vacuum*. However, V admits other vacua states $\phi_0 = \phi_0^a E_0^a$, where the component ϕ_0^a may vanish (i.e., $\phi_0^a = 0$) for some $a \in \{1, \ldots, r\}$. It is an interesting problem, completely open to investigation, to determine vortex configurations asymptotically gauge-equivalent to those vacua, as $|z| \to +\infty$. In this respect, even the extreme case where $\phi_0^a = 0$, $\forall a = 1, \ldots, r$, (known as the *unbroken vacuum state*) describing the "non-topological" situation, has not been handled yet.

On the contrary, for the Maxwell–Chern–Simons–Higgs model, non-topological vortex solutions satisfying the boundary condition (3.5.4) are available. They have been constructed by Chae–Imanuvilov in [ChI3] by extending to the elliptic system (2.1.14) the perturbation approach in [ChI1] discussed above.

However, nothing is known on whether they admit a "concentration" behavior around the vortex points (as the Chern–Simons parameter tends to zero), an important information for meaningful physical applications.

Finally, about the planar topological vortex configurations discussed above (as for the 6th-order Chern–Simons model), it is reasonable to expect their unique (and possibly smooth) dependence on the vortex points. But for both the Maxwell–Chern–Simons–Higgs model and the non-abelian Chern–Simons model, such a uniqueness issue is still unresolved.

4

Periodic Selfdual Chern–Simons Vortices

4.1 Preliminaries

We devote this Chapter to the study of periodic Chern–Simons vortices. Again, we concentrate mainly on the 6th-order Chern–Simons model (1.2.45), where we can carry out a rather complete analysis. We shall give indications of a possible generalization to other models.

Let $\Omega \subset \mathbb{R}^2$ be the periodic cell domain in (2.1.27), and $\{z_1, \ldots, z_N\} \subset \Omega$ be the assigned vortex points (repeated according to their multiplicity). By considering the normalization $\nu^2 = 1$ (always possible via (3.1.2)), we are lead to investigate the following elliptic problem

$$\begin{cases} -\Delta u = \lambda e^u (1 - e^u) - 4\pi \sum_{j=1}^{N} \delta_{z_j} \text{ in } \Omega, \\ u \text{ doubly periodic on } \partial\Omega, \end{cases} \tag{4.1.1}$$

with $\lambda > 0$ given in (3.1.3).

Again, it is convenient to work with v, the regular part of u, defined via the decomposition

$$u = u_0 + v, \tag{4.1.2}$$

with u_0 the *unique* solution (see [Au]) of the problem

$$\begin{cases} \Delta u_0 = 4\pi \sum_{j=1}^{N} \delta_{z_j} - \frac{4\pi N}{|\Omega|} \text{ in } \Omega, \\ \int_\Omega u_0 = 0, \ u_0 \text{ doubly periodic on } \partial\Omega, \end{cases} \tag{4.1.3}$$

with $|\Omega| =$ Lebesgue measure on Ω.

Thus, in terms of v, the problem is reduced to solving:

$$\begin{cases} -\Delta v = \lambda e^{u_0+v} (1 - e^{u_0+v}) - \frac{4\pi N}{|\Omega|} \text{ in } \Omega, \\ v \text{ doubly periodic on } \partial\Omega. \end{cases} \tag{4.1.4}$$

Notice that the weight function e^{u_0} is smooth, doubly periodic on $\partial\Omega$, and vanishes exactly at $z_1, \dots, z_N$ according to their multiplicity. More precisely,

$$e^{u_0(z)} = O\left(|z - z_j|^{2n_j}\right), \text{ as } z \to z_j, \tag{4.1.5}$$

for n_j the multiplicity of z_j and $j = 1, \dots, N$. We start by pointing out some elementary properties valid for a solution v of (4.1.4).

First of all, as an immediate consequence of the maximum principle, we have

$$u_0 + v < 0 \text{ in } \Omega; \tag{4.1.6}$$

while, after integration over Ω, we find

$$\lambda \int_\Omega e^{u_0+v}\left(1 - e^{u_0+v}\right) = 4\pi N. \tag{4.1.7}$$

From (4.1.7) we see that

$$\lambda \int_\Omega \left(e^{u_0+v} - \frac{1}{2}\right)^2 = \frac{\lambda}{4}|\Omega| - 4\pi N.$$

And so we deduce the following *necessary* condition for the solvability of (4.1.4):

$$\lambda \geq \frac{16\pi N}{|\Omega|}. \tag{4.1.8}$$

In addition, if we write

$$v = w + d, \text{ with } \int_\Omega w = 0 \text{ and } d = \fint_\Omega v; \tag{4.1.9}$$

then by (4.1.6) and (4.1.7), we find

$$\frac{4\pi N}{\lambda \int_\Omega e^{u_0+w}} \leq e^d \leq \frac{|\Omega|}{\int_\Omega e^{u_0+w}}, \tag{4.1.10}$$

$$||\Delta w||_{L^\infty(\Omega)} = ||\Delta v||_{L^\infty(\Omega)} \leq \lambda + \frac{4\pi N}{|\Omega|}.$$

Therefore, well-known elliptic estimates imply that w is bounded in $\overline{\Omega}$, and uniformly so, for every λ in a bounded subset of the interval $[\frac{16\pi N}{|\Omega|}, +\infty)$. Since Jensen's inequality (see (2.5.8)) implies that $\fint_\Omega e^{u_0+w} \geq 1$, we see that the same uniform boundedness property holds for v. In fact, after a bootstrap argument, such a property can be extended to hold in $C^m(\overline{\Omega})$-norm, for any given $m \in \mathbb{Z}^+$.

In other words the following holds:

Lemma 4.1.1 *(a) If problem (4.1.4) admits a solution, then* $\lambda \geq \frac{16\pi N}{|\Omega|}$.
(b) For a given $\lambda_0 > \frac{16\pi N}{|\Omega|}$*, the set of solutions of (4.1.4) with* $\lambda \in [0, \lambda_0]$ *is* compact *in* $C^{2,\alpha}(\overline{\Omega})$ $\alpha \in (0, 1)$*, and in any other relevant space.*

In the following we shall be interested in describing the asymptotic behavior of the solution for (4.1.4) as $\lambda \to +\infty$. For this reason, we point out also the following estimate (uniform in λ) in the weaker norm.

Lemma 4.1.2 *Let $\lambda > 0$ and v be a solution for (4.1.4). For any $q \in (1,2)$ there exists a constant $C_q > 0$ (independent of λ) such that*

$$||\nabla v||_{L^q(\Omega)} \leq C_q.$$

Remark 4.1.3 As a consequence of Lemma 4.1.2 and Sobolev's embedding theorem, we know that, for *any* sequence $\lambda_n \to +\infty$, if v_{λ_n} solves (4.1.4) with $\lambda = \lambda_n$, then (along a subsequence) $w_n = v_{\lambda_n} - ⨍_\Omega v_{\lambda_n}$ converges in $L^p(\Omega)$, $\forall p \geq 1$ and pointwise almost everywhere.

Proof of Lemma 4.1.2. Set $w = v - ⨍_\Omega v$, and for $q \in (1,2)$ let $p = \frac{q}{q-1}$ be its dual exponent. We consider the Sobolev space

$$\mathcal{H}_p(\Omega) = \left\{ v \in W^{1,p}_{\text{loc}}(\mathbb{R}^2) : v \text{ doubly periodic, with periodic cell domain } \Omega \right\} \tag{4.1.11}$$

$$= W^{1,p}\left(\mathbb{R}^2 \big/ \mathbf{a}_1\mathbb{Z} \times \mathbf{a}_2\mathbb{Z}\right);$$

and recall that

$$\begin{aligned} ||\nabla v||_{L^q(\Omega)} &= ||\nabla w||_{L^q(\Omega)} \\ &= \sup\left\{ \int_\Omega \nabla w \nabla \phi : \ \phi \in \mathcal{H}_p(\Omega)\ ||\nabla \phi||_{L^p(\Omega)} = 1 \text{ and } \int_\Omega \phi = 0 \right\}. \end{aligned}$$

Since for $p > 2$, we have the continuous Sobolev embedding: $\mathcal{H}_p(\Omega) \hookrightarrow L^\infty(\Omega)$, then for $\phi \in \mathcal{H}_p$, $\int_\Omega \phi = 0$ and $||\nabla \phi||_{L^p(\Omega)} = 1$, we find $||\phi||_{L^\infty(\Omega)} \leq C$ for a suitable constant $C > 0$.

So, by means of (4.1.6) and (4.1.7), we can estimate

$$\begin{aligned} \int_\Omega \nabla w \cdot \nabla \phi &= -\int_\Omega \Delta w \cdot \phi = -\int_\Omega \Delta v \cdot \phi = \lambda \int_\Omega e^{u_0+v}(1 - e^{u_0+v})\phi \\ &\leq ||\phi||_{L^\infty(\Omega)} \lambda \int_\Omega e^{u_0+v}(1 - e^{u_0+v}) \leq 4\pi N C, \end{aligned}$$

and the desired estimate follows. □

Concerning the structure of the solution-set of (4.1.4) (for $\lambda > 0$ large), we see that by the condition (4.1.7), we are still lead to expect *two* classes of solutions. Namely, those satisfying

$$e^{u_0+v} \to 1 \text{ a.e. in } \Omega, \text{ as } \lambda \to +\infty, \tag{4.1.12}$$

which, in analogy to the planar case, we shall call of the "topological-type"; and those satisfying

$$e^{u_0+v} \to 0 \text{ a.e. in } \Omega, \text{ as } \lambda \to +\infty, \tag{4.1.13}$$

which we call of the "non-topological-type."

We devote the rest of this chapter to constructing solutions to (4.1.4) for each such type.

4.2 Construction of periodic "topological-type" solutions

We take $p = 2$ in (4.1.11), then $\mathcal{H}_{p=2}(\Omega)$ reduces to the Sobolev space $\mathcal{H}(\Omega)$ defined in (2.4.23), and furnishes the natural space in which to seek solutions for (4.1.4).

A first contribution towards the existence of solutions for (4.1.4) (for $\lambda > 0$ large) was provided by Caffarelli–Yang in [CY] by means of a sub/supersolution method.

To this purpose, recall that a function $v \in \mathcal{H}(\Omega)$ is called a (weak) subsolution (respectively supersolution) for (4.1.4) if $\forall \varphi \in \mathcal{H}(\Omega)$, with $\varphi \geq 0$ in Ω we have:

$$\int_\Omega \nabla v \cdot \nabla \varphi - \lambda \int_\Omega e^{u_0+v}(1 - e^{u_0+v})\varphi \leq 0 \quad (\text{resp. } \geq 0). \tag{4.2.1}$$

Lemma 4.2.4 *There exists $\lambda_* > 0$ and a smooth function $v_- \in \mathcal{H}(\Omega)$ such that v_- defines a subsolution for (4.1.4), $\forall \lambda > \lambda_*$.*

Proof. We shall construct v_- as a smooth doubly periodic function on $\partial\Omega$ satisfying

$$-\Delta v \leq \lambda e^{u_0+v}(1 - e^{u_0+v}) - \frac{4\pi N}{|\Omega|} \text{ in } \Omega, \tag{4.2.2}$$

for large values of λ.

To this purpose, let $\varepsilon > 0$ be sufficiently small so that $\overline{B_{2\varepsilon}(z_j)} \subset \Omega$, $j = 1, \ldots, N$ and for $z_j \neq z_k$, $B_{2\varepsilon}(z_j) \cap B_{2\varepsilon}(z_k) = \emptyset$. Consider a smooth cut-off function $f_\varepsilon \in C_0^\infty(\Omega)$ such that $0 \leq f_\varepsilon \leq 1$, $f_\varepsilon = 1$ in $\cup_{j=1}^N B_\varepsilon(z_j)$ while $f_\varepsilon(z) = 0$, $\forall z \notin \cup_{j=1}^N B_{2\varepsilon}(z_j)$. Set

$$g_\varepsilon = \frac{8\pi N}{|\Omega|}\left(f_\varepsilon - \fint_\Omega f_\varepsilon\right).$$

Let v_-^0 be the unique solution (depending on ε) for the problem:

$$\begin{cases} \Delta v_-^0 = g_\varepsilon \text{ in } \Omega, \\ v_-^0 \in \mathcal{H}(\Omega) \text{ and } \int_\Omega v_-^0 = 0. \end{cases}$$

Set

$$v_- = v_-^0 - d, \tag{4.2.3}$$

with $d > 0$ sufficiently large to guarantee

$$u_0 + v_- < 0 \text{ in } \Omega. \tag{4.2.4}$$

Note that $\fint_\Omega f_\varepsilon = O(\varepsilon^2)$, and therefore $\forall z \in \cup_{j=1}^N B_\varepsilon(z_j)$, we have:

$$g_\varepsilon(z) = \frac{8\pi N}{|\Omega|}\left(1 + O\left(\varepsilon^2\right)\right) \geq \frac{4\pi N}{|\Omega|}, \text{ for } \varepsilon > 0 \text{ sufficiently small.} \tag{4.2.5}$$

On the other hand, by (4.2.2) it follows that

$$\mu_0^\varepsilon := \max\{e^{u_0+v_-(z)}\left(e^{u_0+v_-(z)} - 1\right), \ \forall z \in \overline{\Omega \setminus \cup_{j=1}^N B_\varepsilon(z_j)}\} < 0,$$

and we find

$$\lambda e^{u_0(z)+v_-(z)}\left(e^{u_0(z)+v_-(z)} - 1\right) + \frac{4\pi N}{|\Omega|} \leq -\lambda|\mu_0^\varepsilon| + \frac{4\pi N}{|\Omega|}, \tag{4.2.6}$$

$\forall\, z \in \Omega \setminus \cup_{j=1}^N B_\varepsilon(z_j)$. Thus, if we combine (4.2.5) and (4.2.6), we see that if we fix $\varepsilon > 0$ sufficiently small, then we find $\lambda_* > 0$ such that

$$\lambda e^{u_0+v_-}\left(e^{u_0+v_-} - 1\right) + \frac{4\pi N}{|\Omega|} \leq g_\varepsilon \text{ in } \Omega, \ \forall \lambda \geq \lambda_*,$$

and (4.2.2) follows with v_- in (4.2.3). □

Remark 4.2.5 By a closer look at the arguments above, we see that the parameter λ_* depends on the value: $\min\limits_{z_j \neq z_k} |z_j - z_k|$, (beside obviously N and Ω) but not on the actual position of the vortex points.

To obtain a solution for (4.1.4), we can proceed either by an iteration scheme (see [CY] and [Y1]), or variationally by means of a minimization problem over $\mathcal{H}(\Omega)$. We shall present the variational approach to remain in the same spirit of the planar topological vortex problem (discussed in Section 3.1). Also, the variational approach will be useful in identifying the second class of "non-topological-type" solutions.

To this purpose, let

$$J_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{\lambda}{2}\int_\Omega \left(e^{u_0+v} - 1\right)^2 + \frac{4\pi N}{|\Omega|}\int_\Omega v, \ v \in \mathcal{H}(\Omega). \tag{4.2.7}$$

By virtue of (2.4.24), we see that J_λ is well-defined, continuosly Fréchet differentiable, and weakly lower semicontinuous in $\mathcal{H}(\Omega)$. Furthermore, its critical points correspond to (weak) solutions to (4.1.4).

Lemma 4.2.6 *Let $v_- \in \mathcal{H}(\Omega)$ be a* subsolution *for (4.1.4) and set*

$$\Lambda = \{v \in \mathcal{H}(\Omega) : \ v \geq v_- \text{ a.e. in } \Omega\}.$$

Then J_λ is bounded from below in Λ where its infimum is attained at a critical point.

Proof. Clearly,

$$\inf_{\Lambda} J_\lambda \geq 4\pi N \fint_\Omega v_-. \tag{4.2.8}$$

To obtain that the infimum in (4.2.8) is achieved, we consider a minimizing sequence $v_n \in \Lambda$; that is

$$J_\lambda(v_n) = \frac{1}{2}\int_\Omega |\nabla v_n|^2 + \frac{\lambda}{2}\int_\Omega \left(e^{u_0+v_n} - 1\right)^2 + \frac{4\pi N}{|\Omega|}\int_\Omega v_n \to \inf_\Lambda J_\lambda.$$

Therefore,

$$||\nabla v_n||_{L^2} \leq C \text{ and } \int_\Omega v_- \leq \int_\Omega v_n \leq C,$$

for a suitable constant $C > 0$. In other words, v_n is uniformly bounded in $\mathcal{H}(\Omega)$, and so we can find a subsequence (denoted in some way):

$$v_n \to v \text{ weak in } \mathcal{H}(\Omega) \text{ and pointwise a.e. in } \Omega.$$

Consequently, $v \in \Lambda$ and by the weakly lower semicontinuity of J_λ, we find

$$J_\lambda(v) = \inf_\lambda J_\lambda.$$

So v gives the desired minimum of J_λ in Λ. The fact that v defines a critical point for J_λ in $\mathcal{H}(\Omega)$ is a consequence of a well-known property of a subsolution (supersolution) in a variational guise (see e.g., Theorem I.24 in [St1] or the appendix of [T1] or Lemma 5.6.3 in [Y1]). □

So, after Lemma 4.2.4 and Lemma 4.2.6, we conclude the existence of a solution for problem (4.1.4), provided that $\lambda > 0$ is sufficiently large. In other words, setting

$$\Gamma = \{\lambda > 0 : \text{ problem (4.1.4) admits a solution}\}, \tag{4.2.9}$$

we then know that Γ contains an infinite interval, and (recalling (4.1.8)), that it is a well-defined value:

$$\lambda_c := \inf\{\lambda,\ \lambda \in \Gamma\} \geq \frac{16\pi N}{|\Omega|}. \tag{4.2.10}$$

Lemma 4.2.7 *We have*

$$\Gamma = [\lambda_c, +\infty), \tag{4.2.11}$$

and so, problem (4.1.4) admits a solution if and only if $\lambda \geq \lambda_c$.

Proof. Let $\lambda_1 \in \Gamma$, so that problem (4.1.4) with $\lambda = \lambda_1$ admits a solution v_1. By virtue of (4.1.6), v_1 defines a *subsolution* for (4.1.4) for every $\lambda > \lambda_1$. Thus, we can apply Lemma 4.2.6 (with v_- repaced by v_1) to obtain a solution of (4.1.4) for every $\lambda > \lambda_1$.

In other words, $\lambda_1 \in \Gamma$ implies that the whole interval $[\lambda_1, +\infty) \subset \Gamma$. This allow us to conclude that $(\lambda_c, +\infty) \subset \Gamma$. Finally, we can also derive a solution for $\lambda = \lambda_c$ by a limiting argument on the basis of Lemma 4.1.1 (b). □

It is useful to observe that for any fixed $\lambda \in [\lambda_c, +\infty)$, by Lemma 4.2.6, we can construct a *maximal* solution $v_{1,\lambda}$ for (4.1.4) just by setting:

$$\begin{aligned} v_{1,\lambda}(z) &= \sup\{v(z) : v \in \mathcal{H}(\Omega) \text{ is a subsolution for (4.1.4)}\} \\ &= \max\{v(z) : v \text{ is a solution for (4.1.4)}\} < -u_0(z). \end{aligned}$$

In particular, $v_{1,\lambda}$ is monotone increasing in λ; that is, if $\mu < \lambda$, then $v_{1,\mu}(z) < v_{1,\lambda}(z)$, $\forall z \in \Omega$. Hence, we can take the pointwise limit as $\lambda \to \lambda_c^+$, and conclude that

$$v_{1,\lambda_c}(z) = \inf_{\lambda > \lambda_c} v_{1,\lambda}(z) = \lim_{\lambda \to \lambda_c^+} v_{1,\lambda}(z).$$

In conclusion, we have proven the following result:

Theorem 4.2.8 *There exist* $\lambda_c \geq \frac{16\pi N}{|\Omega|}$ *such that*

(i) for every $\lambda \geq \lambda_c$*, problem (4.1.4) admits a maximal solution* $v_{1,\lambda}$ *which is monotonically increasing in* λ*. That is,*

$$\textit{if } \lambda_c \leq \lambda < \mu, \textit{ then } v_{1,\lambda}(x) < v_{1,\mu}(x), \ \forall x \in \Omega,$$

and satisfies

$$u_0 + v_{1,\lambda}(x) < 0, \ \forall x \in \Omega.$$

In particular,

$$v_{1,\lambda_c}(x) = \inf_{\lambda > \lambda_c} v_{1,\lambda}(x), \ \forall x \in \Omega.$$

(ii) For $\lambda < \lambda_c$*, problem (4.1.4) admits no solutions.*

Remark 4.2.9 Concerning the dependence of λ_c on the position of vortex points $z_1, \ldots, z_N$, we can take into account Remark 4.2.5 to see that besides Ω and N, its value depends only on (the reciprocal of) the minimal distance between distinct vortex points and not on their actual locations. So a "discontinuity" for $\lambda_c(z_1, \ldots, z_N)$ can occur only when different vortices collapse together.

Setting

$$S_\lambda = \{v \in \mathcal{H}(\Omega) : v \text{ is a solution for (4.1.4)}\}, \tag{4.2.12}$$

we have then shown that the set $S_\lambda \neq \emptyset$ if and only if $\lambda \geq \lambda_c$, in which case it always admits a maximal element.

Remark 4.2.10 For $\lambda > \lambda_c$, it is also possible to construct a *minimal* solution ω_λ among all possible solutions of (4.1.4) bigger than the subsolution v_{1,λ_c}. To this purpose, it suffices to consider the supersolution,

$$v_\lambda^+(z) = \min\left\{v(z) : v \in S_\lambda \text{ and } v \geq v_{1,\lambda_c} \text{ in } \Omega\right\},$$

and as in Lemma 4.2.6, obtain ω_λ as the minimizer of J_λ on the set $\{v \in \mathcal{H}(\Omega) : v_{1,\lambda_c} \le v \le v_\lambda^+ \text{ a.e. in } \Omega\}$. As a matter of fact, setting

$$\Sigma_\lambda = \{v \in \mathcal{H}(\Omega) : \ v_{1,\lambda_c} \le v \le \omega_\lambda \text{ a.e. in } \Omega\}, \tag{4.2.13}$$

we see that the *minimal* solution ω_λ is characterized by the following properties:

$$J_\lambda(\omega_\lambda) = \inf_{\Sigma_\lambda} J_\lambda, \tag{4.2.14}$$

and

$$S_\lambda \cap \Sigma_\lambda = \{\omega_\lambda\}, \tag{4.2.15}$$

$\forall \lambda > \lambda_c$.

By a recent result of the author (see [T7]), we actually know that the maximal and minimal solution is *one* and the same for sufficiently large $\lambda > 0$.

Next, we exploit further the variational formulation above in order to show that for every $\lambda > \lambda_c$, problem (4.1.4) admits a solution other than the maximal one.

To this purpose, we first need to derive an improvement to the statement of Lemma 4.2.6 based on the observation of Brezis–Nirenberg [BN], asserting that, for general variational principles of elliptic nature, minimization in C^1-norm is equivalent to minimization in H^1-norm.

Let $v_* = v_{1,\lambda_c}$ be the maximal solution for (4.1.4) at $\lambda = \lambda_c$. Since it defines a *strict* subsolution for (4.1.4) for every $\lambda > \lambda_c$, we can use it in Lemma 4.2.6 to obtain a solution v_λ to (4.1.4) satisfying:

$$J_\lambda(v_\lambda) = \inf\{J_\lambda(v) : \ v \in \mathcal{H}(\Omega) \text{ and } v \ge v_*, \text{ a.e. in } \Omega\}. \tag{4.2.16}$$

Note that, by the strong maximum principle, we have the strict pointwise inequality:

$$v_\lambda(x) - v_*(x) > 0, \ \forall x \in \overline{\Omega}. \tag{4.2.17}$$

Therefore, v_λ defines a *local minimum* for J_λ in $C^0(\overline{\Omega})$-topology.

The following stronger result holds:

Lemma 4.2.11 *v_λ defines a* local minimum *for J_λ in $\mathcal{H}(\Omega)$.*

Proof. We argue by contradiction and assume that for every $\rho > 0$ there exists $v \in \mathcal{H}(\Omega)$:

$$||v - v_\lambda|| \le \rho \text{ and } J_\lambda(v) < J_\lambda(v_\lambda).$$

As in the proof of Lemma 4.2.6, we see that J_λ attains its infimum in $\overline{B_\rho(v_\lambda)} = \{v \in \mathcal{H}(\Omega) : \ ||v - v_\lambda|| \le \rho\}$. Hence, for a sequence $\rho_n \to 0$, let v_n represent the corresponding minimum of J_λ in $\overline{B_{\rho_n}(v_\lambda)}$. Therefore $\forall n \in \mathbb{N}$,

$$J_\lambda(v_n) < J_\lambda(v_\lambda), \tag{4.2.18}$$

and as $n \to \infty$,

$$v_n \to v_\lambda \text{ strongly in } \mathcal{H}(\Omega). \tag{4.2.19}$$

In particular,

$$e^{u_0+v_n} \to e^{u_0+v_\lambda} \text{ in } \mathrm{L}^p(\Omega),\ \forall p \geq 1. \tag{4.2.20}$$

Furthermore, we can use the Lagrange multiplier rule to see that v_n satisfies

$$-\Delta v_n + \lambda e^{u_0+v_n}\left(e^{u_0+v_n}-1\right) + \frac{4\pi N}{|\Omega|} = \mu_n\left(-\Delta(v_n - v_\lambda) + v_n - v_\lambda\right),$$

for a suitable Lagrange multiplier $\mu_n \leq 0$. Therefore, if we use (4.1.4) for v_λ, we find that

$$\begin{aligned}-\Delta(v_n - v_\lambda) = {} & \frac{\lambda}{1+|\mu_n|}\left(e^{u_0+v_n}\left(1-e^{u_0+v_n}\right) - e^{u_0+v_\lambda}\left(1-e^{u_0+v_\lambda}\right)\right) \\ & - \frac{|\mu_n|}{1+|\mu_n|}(v_n - v_\lambda).\end{aligned} \tag{4.2.21}$$

By virtue of (4.2.19) and (4.2.20), we see that the right-hand side of (4.2.21) converges to zero in $\mathrm{L}^p(\Omega)$, $\forall p \geq 1$. Consequently, by standard elliptic estimates, we see that $v_n \to v_\lambda$ uniformly in $\overline{\Omega}$. Thus, we can use (4.2.17) to conclude that $v_n > v_*$ in $\overline{\Omega}$, for n sufficiently large. But this is clearly impossible since (4.2.18) would contradict (4.2.16). □

If we combine Lemma 4.2.11 with the fact that the functional J_λ is unbounded from below in $\mathcal{H}(\Omega)$, we see that J_λ admits a "mountain-pass" geometry in the sense of (2.3.3) (cf. [AR]) and we can draw the following conclusion:

Proposition 4.2.12 *For every $\lambda > \lambda_c$, the functional J_λ admits at least two critical points in $\mathcal{H}(\Omega)$.*

As a first step towards the proof of Proposition 4.2.12, we need to establish the following.

Lemma 4.2.13 *The functional J_λ satisfies the (PS)-condition at any level $c \in \mathbb{R}$.*

Proof. Actually, we check the following stronger property: if $\{v_n\} \subset \mathcal{H}(\Omega)$ satisfies $J'_\lambda(v_n) \to 0$ in $\mathcal{H}^*(\Omega)$ ($\simeq \mathcal{H}(\Omega)$), then v_n admits a strongly convergent subsequence.

To this purpose, write

$$v_n = w_n + t_n : \ t_n = \fint_\Omega v_n \text{ and } \int_\Omega w_n = 0.$$

We have

$$\begin{aligned} o(1) &= J'_\lambda(v_n)(1) = \lambda \int_\Omega e^{u_0+v_n}\left(e^{u_0+v_n}-1\right) + 4\pi N, \\ o(||w_n||) &= J'_\lambda(v_n)(w_n) = ||\nabla w_n||_2^2 + \lambda \int_\Omega e^{u_0+v_n}\left(e^{u_0+v_n}-1\right) w_n. \end{aligned} \tag{4.2.22}$$

Using Holder's inequality, from the first property in (4.2.22), we derive

$$\int_\Omega e^{2(u_0+v_n)} - |\Omega|^{\frac{1}{2}} \left(\int_\Omega e^{2(u_0+v_n)} \right)^{\frac{1}{2}} + \frac{4\pi N}{\lambda} \leq o(1),$$

yielding to the uniform estimate

$$\int_\Omega e^{2(u_0+v_n)} \leq C, \tag{4.2.23}$$

for a suitable constant $C > 0$. On the other hand, by means of Jensen's inequality (2.5.8),

$$⨍_\Omega e^{u_0+w_n} \geq \exp\left(⨍_\Omega u_0 + w_n \right) = 1,$$

we can use (4.2.23) to obtain the upper bound:

$$e^{t_n} \leq e^{t_n} ⨍_\Omega e^{u_0+w_n} = ⨍_\Omega e^{u_0+v_n} \leq \left(⨍_\Omega e^{2(u_0+v_n)} \right)^{\frac{1}{2}} \leq C. \tag{4.2.24}$$

At this point, by the second property in (4.2.22), we deduce the estimate:

$$\begin{aligned} o(||w_n||) &= ||\nabla w_n||^2_{\mathrm{L}^2(\Omega)} + \lambda \int_\Omega e^{u_0+v_n}(e^{u_0+v_n} - 1)w_n \\ &= ||\nabla w_n||^2_{\mathrm{L}^2(\Omega)} + \lambda \int_\Omega e^{u_0+v_n}(e^{u_0+w_n+t_n} - e^{u_0+t_n})w_n \\ &\quad + \lambda \int_\Omega e^{u_0+t_n} e^{u_0+v_n} w_n \geq ||\nabla w_n||^2_{\mathrm{L}^2(\Omega)} - C||w_n||_{\mathrm{L}^2(\Omega)}, \ \forall n \in \mathbb{N}. \end{aligned} \tag{4.2.25}$$

Consequently,

$$||\nabla w_n||_{\mathrm{L}^2(\Omega)} \leq C, \tag{4.2.26}$$

for a suitable constant $C > 0$. In particular, by means of Moser–Trudinger inequality (2.4.24), for every $p \geq 1$, we have

$$\left|\left| \int_\Omega e^{u_0+w_n} \right|\right|_{\mathrm{L}^p(\Omega)} \leq C_p, \tag{4.2.27}$$

$\forall n \in \mathbb{N}$, with a suitable constant $C_p > 0$ (depending only on p). Observing that the first identity in (4.2.22) also implies

$$e^{t_n} \int_\Omega e^{u_0+w_n} \geq \frac{4\pi N}{\lambda} + o(1),$$

we can use (4.2.27) (with $p = 1$) to deduce the lower bound

$$t_n \geq \log \frac{4\pi N}{\lambda} - \log C_1 + o(1).$$

Combining the estimates above with (4.2.24), we conclude that

$$|t_n| \leq C, \tag{4.2.28}$$

$\forall n \in \mathbb{N}$ and for a suitable $C > 0$. Since (4.2.26) and (4.2.28) imply that v_n is uniformly bounded in $\mathcal{H}(\Omega)$, we can find a subsequence (denoted in the same way) to assert that

$$v_n \rightharpoonup v \text{ weakly in } \mathcal{H}(\Omega) \text{ and strongly in } \mathrm{L}^2(\Omega),$$

for a suitable $v \in \mathcal{H}(\Omega)$.

Furthermore,

$$\begin{aligned}
||\nabla(v_n - v)||^2_{\mathrm{L}^2} &= J'_\lambda(v_n)(v_n - v) - \lambda \int_\Omega e^{u_0+v_n}(e^{u_0+v_n} - 1)(v_n - v) \\
&\quad -4\pi N \fint_\Omega (v_n - v) + \int_\Omega \nabla v \cdot \nabla(v_n - v) \\
&\leq o(1)||v_n - v|| + \lambda \left(||e^{2(u_0+v_n)}||_{\mathrm{L}^2} + ||e^{(u_0+v_n)}||_{\mathrm{L}^2} + \frac{4\pi N}{|\Omega|^{\frac{1}{2}}} \right) ||v_n - v||_{\mathrm{L}^2(\Omega)} \\
&\quad +o(1) \leq C||v_n - v||_{\mathrm{L}^2(\Omega)} + o(1),
\end{aligned}$$

as follows from (4.2.26), (4.2.27), and (4.2.28). Consequently, $||v_n - v|| \to 0$, and the desired conclusion follows. □

We are now ready to give:

Proof of Proposition 4.2.12. Let $\rho_0 > 0$ be such that

$$J_\lambda(v_\lambda) = \inf_{||v - v_\lambda|| \leq 2\rho_0} J_\lambda(v),$$

with v_λ as given in Lemma 4.2.11. If $J_\lambda(v_\lambda) = \inf_{||v-v_\lambda||=\rho_0} J_\lambda(v)$, then we would find a local minimum for J_λ in $\partial B_{\rho_0}(v_\lambda)$ to provide us with a second critical point. Hence suppose that

$$J_\lambda(v_\lambda) < \inf_{||v-v_\lambda||=\rho_0} J_\lambda(v). \tag{4.2.29}$$

We easily check that $J_\lambda(v_\lambda - t) \to -\infty$ as $t \to +\infty$. So we can find $t_0 > 0$ sufficiently large to satisfy

$$t_0 > \rho_0 \text{ and } J_\lambda(v_\lambda - t_0) < J_\lambda(v_\lambda). \tag{4.2.30}$$

Therefore, by properties (4.2.29) and (4.2.30), we see that J_λ verifies the "mountain-pass" property (2.3.3) with $\Gamma = \partial B_{\rho_0}(v_\lambda)$, $v_1 = v_\lambda$ and $v_2 = v_\lambda - t_0$. Thus, letting

$$\mathcal{P} = \{\gamma : [0, 1] \to \mathcal{H}(\Omega) \text{ continuous, } \gamma(0) = v_\lambda \text{ and } \gamma(1) = v_\lambda - t_0\},$$

and by means of Lemma 4.2.13 and Remark 2.3.6,

$$c_\lambda = \inf_{\gamma \in \mathcal{P}} \max_{t \in [0,1]} J_\lambda(\gamma(t)) > J_\lambda(v_\lambda)$$

defines a critical value for J_λ (see [AR] and [St1] Theorem II.6.1). Since $c_\lambda > J_\lambda(v_\lambda)$, c_λ yields to a critical point different from v_λ. □

As a consequence of Proposition 4.2.12 we have:

Theorem 4.2.14 *For $\lambda > \lambda_c$, problem (4.1.4) admits a second solution (distinct from the maximal solution).*

Our next goal is to obtain more precise information about the asymptotic behavior of those different solutions, as $\lambda \to +\infty$. More precisely, we wish to know whether, respectively, they satisfy the "topological-type" condition of (4.1.12) and the "non-topological-type" condition of (4.1.13). A criterion that distinguishes between these two classes of solutions is given as follows:

Lemma 4.2.15 *Let v_λ be a solution to (4.1.4), and set*

$$d_\lambda = \fint_\Omega v_\lambda.$$

(1) If $\limsup_{\lambda \to +\infty} \lambda e^{d_\lambda} < +\infty$*, as* $\lambda \to +\infty$*, then*

$$e^{u_0+v_\lambda} \to 0, \text{ in } \mathrm{L}^p(\Omega)\ \forall p \geq 1 \text{ and pointwise a.e. in } \Omega. \tag{4.2.31}$$

(2) If $\lambda e^{d_\lambda} \to +\infty$, as $\lambda \to +\infty$ (or possibly along a sequence $\lambda_n \to +\infty$), then as $\lambda \to +\infty$ (or along the given sequence):

$$(i)\quad u_0 + v_\lambda \to 0 \text{ in } \mathrm{L}^p(\Omega) \cap C^m_{loc}(\overline{\Omega} \setminus \{z_1, \ldots, z_N\}), \tag{4.2.32}$$
$$\text{for every } p \geq 1, \text{ and } m \in \mathbb{Z}^+;$$

$$(ii)\quad \lambda e^{u_0+v_\lambda}(1 - e^{u_0+v_\lambda}) \to 4\pi \sum_{j=1}^{N} \delta_{z_j} = 4\pi \sum_{j \in J} n_j \delta_{z_j}, \tag{4.2.33}$$

$$\lambda(1 - e^{u_\lambda})^2 \to 4\pi \sum_{j \in J} n_j^2 \delta_{z_j}, \tag{4.2.34}$$

$$\lambda(1 - e^{u_\lambda}) \to 4\pi \sum_{j \in J} n_j(1 + n_j)\delta_{z_j}, \tag{4.2.35}$$

weakly in the sense of measure in $\overline{\Omega}$. Where $J \subset \{1, \ldots, N\}$ is a set of indices identifying all distinct vortices in $\{z_1, \ldots, z_N\}$, and $n_j \in \mathbb{N}$ is the multiplicity of z_j for $j \in J$.

Proof. To establish (4.2.31), notice that by the given assumption, necessarily $d_\lambda \to -\infty$ as $\lambda \to +\infty$. As a consequence, we claim that

$$e^{u_0+v_\lambda} \to 0 \quad \text{pointwise a.e. in } \Omega. \tag{4.2.36}$$

Indeed, on the basis of Remark 4.1.3 we know that, for any sequence $\lambda_n \to +\infty$,

$$w_n = v_{\lambda_n} - d_{\lambda_n} = v_{\lambda_n} - \fint v_{\lambda_n} \to w_0, \text{ in } \mathrm{L}^p(\Omega) \text{ and pointwise a.e. in } \Omega$$

(by taking a subsequence if necessary). Consequently, $e^{u_0+v_{\lambda_n}} = e^{d_n}e^{u_0+w_n} \to 0$ pointwise a.e. in Ω, for *any* sequence $\lambda_n \to +\infty$. Thus we deduce (4.2.36). At this point by (4.1.6), we can use dominated convergence to conclude that $e^{u_0+v_\lambda} \to 0$ in $\mathrm{L}^p(\Omega), \forall p \geq 1$ as claimed.

To establish (4.2.32) and (4.2.36), we start by observing that the given assumption implies that,

$$d_\lambda \to 0 \text{ and } u_0 + v_\lambda \to 0 \text{ in } \mathrm{L}^p(\Omega), \;\; p \geq 1; \tag{4.2.37}$$

as $\lambda \to +\infty$ (or along the given sequence $\lambda_n \to +\infty$). To this purpose, notice that by (4.1.6) we have $e^{d_\lambda} \in (0, 1), \;\forall \lambda > 0$. Assume that

$$\liminf_{\lambda \to +\infty} e^{d_\lambda} := A \in [0, 1]. \tag{4.2.38}$$

Then along a sequence $\lambda_k \to +\infty$, we find

$$e^{d_{\lambda_k}} \to A \text{ and } w_k = v_{\lambda_k} - \fint v_{\lambda_k} \to w_0 \text{ in } \mathrm{L}^p(\Omega)\; \forall p \geq 1, \text{ and pointwise a.e. in } \Omega.$$

Notice, in particular, that

$$\int_\Omega w_0 = 0. \tag{4.2.39}$$

On the other hand, if we use (4.1.7), then

$$\int_\Omega e^{u_0+w_k}(1 - e^{u_0+v_{\lambda_k}}) = \frac{4\pi N}{\lambda_k e^{d_{\lambda_k}}} \to 0, \text{ as } k \to +\infty.$$

So we can use Fatou's lemma to conclude that $Ae^{u_0+w_0} = 1$. Hence $A > 0$, and since

$$u_0 + w_0 = \log \frac{1}{A},$$

we can use (4.2.39) to find

$$\log \frac{1}{A} = \fint_\Omega u_0 + w_0 = 0, \text{ that is } A = 1.$$

Consequently $d_\lambda \to 0$, as $\lambda \to +\infty$ (or along the given sequence), and $u_0 + v_\lambda \to 0$ in $\mathrm{L}^p(\Omega), \forall p \geq 1$.

With this information, we can proceed exactly as in the proof of Proposition 3.2.9, to obtain the estimates

$$\left|\lambda \int_{\Omega} e^{u_0+v_\lambda}(1-e^{u_0+v_\lambda})\varphi - 4\pi \sum_{j=1}^{N} \varphi(z_j)\right| \le ||\Delta\varphi||_{L^\infty(\Omega)}||u_0+v_\lambda||_{L^1(\Omega)} \quad (4.2.40)$$

$\forall \varphi \in \mathcal{H}(\Omega) \cap C^\infty$, and for $\Omega_\delta = \Omega \setminus \cup_{j=1}^N B_\delta(z_j),\ \delta > 0$,

$$||u_0+v_\lambda||_{L^\infty(\Omega_\delta)} + ||\lambda e^{u_0+v_\lambda}(1-e^{u_0+v_\lambda})||_{L^\infty(\Omega_\delta)} \le \frac{C_\delta(p)}{\lambda} \quad (4.2.41)$$

for any given $p \ge 1$, with $C_\delta(p) > 0$ a suitable constant independent of λ.

Thus, from (4.2.40) we readily get (4.2.33); while we can complete the proof of (4.2.32), by virtue of (4.2.41) and a bootstrap argument. Finally, by arguing as in the proof of Corollary 3.2.10, we can use a Pohozaev-type identity around each vortex point $z_j,\ j = 1, \dots, N$ to deduce (4.2.34) and (4.2.35). □

By Lemma 4.2.15 we can immediately classify the *maximal* solution $v_{1,\lambda}$, as being of the "topological-type." Indeed,

$$\text{for } \lambda \ge \lambda_c,\ d_{1,\lambda_c} = \fint_{\Omega} v_{1,\lambda_c} \le \fint_{\Omega} v_{1,\lambda} := d_{1,\lambda};$$

and therefore, $\lambda e^{d_{1,\lambda}} \to +\infty$. Thus we conclude:

Corollary 4.2.16 *The maximal solution of (4.1.4) (as given by Theorem 4.2.8) satisfies the "topological-type" condition (4.1.12), where the convergence holds in* $L^p(\Omega)$ $\forall p \ge 1$ *and pointwise a.e. in* Ω. *In addition, it satisfies the convergence properties (4.2.32), (4.2.33), (4.2.34), and (4.2.35).*

4.3 Construction of periodic "non-topological-type" solutions

At this point, it is natural to ask whether the "mountain-pass" solution constructed above is of the "non-topological-type." A first answer to this question has been provided by Ding–Jost–Li–Pi–Wang in [DJLPW] with an approach that works equally well in a higher dimension and allows us to treat vortex-type solutions for the Sieberg–Witten functional (cf. [Jo]). In our context their result completes Theorem 4.2.14 as follows:

Theorem 4.3.17 *[DJLPW] For* $\lambda > 0$ *sufficiently large, problem (4.1.4) admits a "non-topological-type" solution* v_λ, *in the sense that (4.2.31) holds.*

Before going into the details of the proof of Theorem 4.3.17, we mention that previous results relative to "non-topological type" solutions are contained in [T1], [DJLW2], [DJLW3], and [NT3]. Those results concern mainly the single or double vortex case (i.e., $N = 1, 2$), and have the advantage to yield solutions that verify the convergence property in (4.2.31) with respect to $C^0(\Omega)-$norm. We shall return to discuss this aspect in Section 4.4.

As already mentioned, the proof of Theorem 4.3.17 relies in a "mountain-pass" construction. However to attain the desired non-topological information, we need to replace the local minimum v_λ of Lemma 4.2.11 with the *minimal* solution ω_λ, as characterized by (4.2.14) and (4.2.15). But we can check that ω_λ defines a local minimum only with respect to the stronger $W^{2,2}\left(\mathbb{R}^2 \diagup \mathbf{a}_1\mathbb{Z} \times \mathbf{a}_2\mathbb{Z}\right)$-topology. Therefore, we also need to modify the pseudogradient deformation flow in (2.3.6), (2.3.7) (cf. [St1], [R]) with the more regular heat flow:

$$\begin{cases} v_t = \Delta v + \lambda e^{u_0+v}(1 - e^{u_0+v}) - \frac{4\pi N}{|\Omega|} \text{ in } \Omega \times \mathbb{R}^+, \\ v(\cdot, 0) = g_0, \\ v(\cdot, t) \text{ doubly periodic on } \partial\Omega, \ \forall t \geq 0, \end{cases} \tag{4.3.1}$$

where the initial data g_0 is suitably chosen.

To be more precise, let

$$X_\lambda = \left\{ v \in W^{2,2}\left(\mathbb{R}^2 \diagup \mathbf{a}_1\mathbb{Z} \times \mathbf{a}_2\mathbb{Z}\right) : \ v \leq \omega_\lambda \text{ in } \Omega \right\} \subset \mathcal{H}(\Omega).$$

Then X_λ is a closed convex subset of $W^{2,2}\left(\mathbb{R}^2 \diagup \mathbf{a}_1\mathbb{Z} \times \mathbf{a}_2\mathbb{Z}\right)$, with induced norm denoted by $||\cdot||_{X_\lambda}$.

Since ω_λ is a solution for (4.1.4), we know that X_λ is invariant under the heat flow (4.3.1), and the following holds:

Lemma 4.3.18 *For every $g_0 \in X_\lambda$, problem (4.3.1) admits a unique smooth solution v in $\Omega \times (0, +\infty)$, such that $v(\cdot, t) \in X_\lambda$, which depends continuously on the initial data g_0, $\forall t \in (0, +\infty)$. Furthermore:*

(i) The map: $t \to (||v(\cdot, t)||_{X_\lambda}, \ ||v_t(\cdot, t)||_{L^2(\Omega)})$ is continuous in $[0, +\infty)$ and $||v(\cdot, t) - g_0||_{X_\lambda} \to 0$, as $t \to 0^+$.
(ii) If

$$\sup_{t>0} ||v(\cdot, t)||_{L^2(\Omega)} + ||v_t(\cdot, t)||_{L^2(\Omega)} < +\infty, \tag{4.3.2}$$

then as $t \to +\infty$, $v(\cdot, t)$ converges in X_λ to a solution for (4.1.4).

Proof. By general local existence results for nonlinear parabolic equations (cf. [LSU] and [Fre]), we know that for $g_0 \in X_\lambda$ there exists $T > 0$ such that (4.3.1) admits a solution $v \in L^2(0, T; X_\lambda) \cap L^\infty(0, T; \mathcal{H}(\Omega))$ and $v_t \in L^2(0, T; L^2(\Omega))$. Since $v(\cdot, t) \in X_\lambda$, then $u_0(z) + v(z, t) < 0$, $\forall z \in \Omega$ and $\forall t \in [0, T]$. Therefore, setting

$$f(z, t) = \lambda e^{u_0(z)+v(z,t)} \left(1 - e^{u_0(z)+v(z,t)}\right) - \frac{4\pi N}{|\Omega|}, \tag{4.3.3}$$

we find

$$||f||_{L^\infty(\Omega\times[0,T])} \leq \lambda + \frac{4\pi N}{|\Omega|} \text{ and } \left|\frac{\partial f}{\partial t}\right| \leq 3\lambda|v_t| \text{ in } \Omega \times [0, T]. \tag{4.3.4}$$

By familiar arguments, we derive the estimates:

$$\frac{1}{2}\frac{d}{dt}||v_t(\cdot,t)||^2_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2}}\left(\lambda + \frac{4\pi N}{|\Omega|}\right)||v(\cdot,t)||_{L^2(\Omega)} \tag{4.3.5}$$

$$\frac{1}{2}\frac{d}{dt}||v(\cdot,t)||^2_{L^2(\Omega)} \leq 3\lambda||v_t(\cdot,t)||^2_{L^2(\Omega)}, \ \forall t \in [0,T]. \tag{4.3.6}$$

Thus, by means of a Gronwall type inequality, from (4.3.5) and (4.3.6), we deduce:

$$||v_t(\cdot,t)||^2_{L^2(\Omega)} + ||v(\cdot,t)||^2_{L^2(\Omega)} \leq c_\lambda\left(1 + e^{3\lambda t}\right)||g_0||_{X_\lambda}, \ \forall t \in [0,T], \tag{4.3.7}$$

with $c_\lambda > 0$ a constant independent of t. In turn, by elliptic estimates, we find

$$||\nabla v_t(\cdot,t)||^2_{L^2(\Omega)} + ||v(\cdot,t)||_{X_\lambda} \leq C_\lambda\left(1 + e^{3\lambda t}\right)||g_0||_{X_\lambda}, \ \forall t \in [0,T], \tag{4.3.8}$$

with C_λ a suitable constant depending on λ only.

In particular, $v_t \in L^2(0,T;\mathcal{H}(\Omega))$ and we deduce (i) by means of well-known results (see e.g., [Ev] and [LSU]). Furthermore, we can use a bootstrap argument together with standard parabolic estimates to derive estimates in Holder spaces analogous to (4.3.8). In this way, we can check for smoothness of the solution and use a continuation argument to conclude that necessarily $T = +\infty$. So v is globally defined and smooth in $\Omega \times (0,+\infty)$ as claimed. Moreover if (4.3.2) holds, then we can replace the right-hand side of (4.3.7) with a constant independent of t, and in turn obtain corresponding estimates for v and v_t in Holder spaces to hold uniformly, $\forall t > 0$. This fact allows us to show convergence of $v(\cdot,t)$ in X_λ, as $t \to +\infty$, towards a limiting function that must satisfy (4.1.4).

Similarly, we can estimate the difference between two solutions v_1 and v_2 satisfying (4.3.1) in terms of their initial data g_1 and $g_2 \in X_\lambda$, respectively. Thus, we find that

$$||v_1(\cdot,t) - v_2(\cdot,t)||_{X_\lambda} \leq C_\lambda\left(e^{3\lambda t} + 1\right)||g_1 - g_2||_{X_\lambda}, \ \forall t > 0, \tag{4.3.9}$$

for suitable $C_\lambda > 0$ depending on λ only.

This proves uniqueness and continuous dependence of the solution on the initial data. □

On the basis of the information in Lemma 4.3.18 we obtain:

Lemma 4.3.19 *The minimal solution ω_λ defines a strict local minimum for J_λ in X_λ.*

Proof. First of all observe that, if $v = v(z,t)$ is a solution of (4.1.4), then the functional $J_\lambda(v(\cdot,t))$ is monotone decreasing in t, and we have

$$\frac{d}{dt}J_\lambda(v(\cdot,t)) = -\int_\Omega |v_t(x,t)|^2\,dx = -||v_t(\cdot,t)||^2_{L^2(\Omega)}. \tag{4.3.10}$$

We argue by contradiction and suppose that for every $\delta > 0$ small, we have

$$w_\delta \in X_\lambda : \ ||\omega_\lambda - w_\delta||_{X_\lambda} = \delta \text{ and } J_\lambda(w_\delta) \le J_\lambda(\omega_\lambda). \tag{4.3.11}$$

Recall that $\omega_\lambda - v_{1,\lambda_c} > 0$ in $\overline{\Omega}$. So for $\delta > 0$ sufficiently small, we can use Lemma 4.3.18, to find $T_\delta > 0$ small, such that, if v is the solution of (4.3.1) with $v(\cdot, 0) = w_\delta$, then

$$v_{1,\lambda_c} < v(\cdot, t) \le \omega_\lambda \ \text{ in } \ \overline{\Omega} \ \text{ and } \ ||v(\cdot, t) - \omega_\lambda||_{X_\lambda} \ge \frac{\delta}{2},$$

for any $t \in [0, T_\delta]$.

In view of (4.2.12), (4.2.13), and (4.2.15), we find that $v(\cdot, t) \in \Sigma_\lambda$, but $v(\cdot, t) \notin S_\lambda$, $\forall t \in [0, T_\delta]$. Consequently,

$$\varepsilon_0 := \min_{t \in [0, T_\delta]} ||v_t(\cdot, t)||^2_{L^2(\Omega)} > 0,$$

and by (4.3.10) and (4.3.11) we obtain

$$J_\lambda(v(\cdot, T_\delta)) \le J_\lambda(w_\delta) - \int_0^{T_\delta} ||v_t(\cdot, t)||^2_{L^2(\Omega)} \le J_\lambda(\omega_\lambda) - \varepsilon_0 T_\delta < J_\lambda(\omega_\lambda),$$

in contradiction to (4.2.14). □

Proof of Theorem 4.3.17. As a consequence of Lemma 4.3.19, we find $\rho_0 > 0$ and $\varepsilon_0 > 0$ such that

$$\forall v \in X_\lambda : \ ||v - \omega_\lambda||_{X_\lambda} = \rho_0 \text{ then } J_\lambda(v) \ge J_\lambda(\omega_\lambda) + \varepsilon_0. \tag{4.3.12}$$

Since $J_\lambda(\omega_\lambda - s) \to -\infty$ as $s \to +\infty$, we can fix $s_0 > 0$ sufficiently large so that

$$J_\lambda(\omega_\lambda - s_0) \le \min_{||v - \omega_\lambda||_{X_\lambda} \le \rho_0} J_\lambda - 1. \tag{4.3.13}$$

For any $s \in [0, s_0]$ and $t \ge 0$, we consider the two-parameter family of functions $v(z, t, s)$ such that $v(\cdot, \cdot, s)$ is the unique solution for (4.3.1) with $v(\cdot, 0, s) = \omega_\lambda - s$. Observe that

$$v(\cdot, t, 0) = \omega_\lambda, \ \forall t \ge 0,$$

and by (4.3.10),

$$J_\lambda(v(\cdot, t, s_0)) \le J_\lambda(v(\cdot, 0, s_0)) = J_\lambda(\omega_\lambda - s_0).$$

Therefore, by (4.3.13), the function $v(\cdot, t, s_0)$ must lie outside the ball in X_λ with center ω_λ and radius ρ_0. Hence, by a continuity argument, $\forall t > 0$ there must exist $s_t \in (0, s_0)$ such that

$$||v(\cdot, t, s_t) - \omega_\lambda||_{X_\lambda} = \rho_0,$$

and so,

$$J_\lambda(v(\cdot, t, s_t)) \ge J_\lambda(\omega_\lambda) + \varepsilon_0.$$

Now, along a sequence $t_n \to +\infty$ we may suppose that $s_{t_n} \to s_*$, and by the monotonicity property (4.3.10), we find

$$J_\lambda(v(\cdot, t, s_{t_n})) \geq J_\lambda(v(\cdot, t_n, s_{t_n})) \geq J_\lambda(\omega_\lambda) + \varepsilon_0,$$

for any $t \in (0, +\infty)$ and n sufficiently large. Passing to the limit as $n \to +\infty$, we conclude:

$$J_\lambda(v(\cdot, t, s_*)) \geq J_\lambda(\omega_\lambda) + \varepsilon_0, \ \forall t \in [0, +\infty). \tag{4.3.14}$$

This implies that $s_* > 0$ and

$$\int_0^t ||v_t(\cdot, \tau, s_*)||^2_{L^2(\Omega)} \, d\tau = J_\lambda(\omega_\lambda - s_*) - J_\lambda(v(\cdot, t, s_*)) \leq C_1, \ \forall t > 0; \tag{4.3.15}$$

with a suitable constant $C_1 > 0$ independent of t. Consequently, we can use (4.3.6) to deduce

$$\begin{aligned} ||v_t(\cdot, t, s_*)||^2_{L^2(\Omega)} &\leq ||\omega_\lambda - s_*||_{X_\lambda} + \int_0^t ||v_t(\cdot, \tau, s_*)||^2_{L^2(\Omega)} \, d\tau \\ &\leq C_2, \ \forall t > 0, \end{aligned} \tag{4.3.16}$$

with a suitable constant $C_2 > 0$ independent of $t \in [0, +\infty)$. At this point, by means of the equation (4.3.1), we find

$$\begin{aligned} ||\nabla v(\cdot, t, s_*)||^2_{L^2(\Omega)} &= \int_\Omega \left(\lambda e^{u_0+v}\left(1 - e^{u_0+v}\right) - v_t\right)\left(v - ⨍_\Omega v\right) \\ &\leq (\lambda|\Omega|^{\frac{1}{2}} + C_2)||v - ⨍_\Omega v||_{L^2(\Omega)}, \end{aligned}$$

and by Poincare's inequality we derive the uniform estimate

$$||\nabla v(\cdot, t, s_*)||_{L^2(\Omega)} \leq C_3, \ \forall t > 0 \tag{4.3.17}$$

with a suitable constant $C_3 > 0$, independent of t.

Moreover,

$$\begin{aligned} ⨍_\Omega v(\cdot, t, s_*) &= \frac{1}{4\pi N}\left(J_\lambda(v(\cdot, t, s_*)) - \frac{1}{2}||\nabla v(\cdot, t, s_*)||^2_{L^2(\Omega)} - \frac{\lambda}{2}\int_\Omega \left(1 - e^{u_0+v_\lambda}\right)^2\right) \\ &\geq \frac{1}{4\pi N}(J_\lambda(\omega_\lambda) + \varepsilon_0 - \frac{1}{2}C_3^2 - \frac{\lambda}{2}|\Omega|); \end{aligned}$$

and recalling that $v(\cdot, t, s_*) < \omega_\lambda$ in Ω, we obtain

$$\left| ⨍_\Omega v(\cdot, t, s_*) \right| \leq C, \ \forall t > 0; \tag{4.3.18}$$

with a suitable constant C independent of t. So, by combining (4.3.16), (4.3.17), and (4.3.18), we check the validity of assumption (4.3.2) for $v(\cdot, t, s_*)$. Therefore, by letting $t \to +\infty$, we conclude that

$$v(\cdot, t, s_*) \to v_{2,\lambda} \text{ in } X_\lambda$$

(and in any other relevant norm) with $v_{2,\lambda}$ a solution for (4.1.4) which satisfies

$$v_{2,\lambda} \leq \omega_\lambda \text{ in } \Omega. \tag{4.3.19}$$

Since

$$J_\lambda(v_{2,\lambda}) = \lim_{t\to+\infty} J_\lambda(v(\cdot, t, s_*)) \geq J_\lambda(\omega_\lambda) + \varepsilon_0,$$

we see that necessarily $v_{2,\lambda} \neq \omega_\lambda$, and so (4.3.19) holds with a strict inequality. Furthermore, property (4.2.15) implies that

$$v_{2,\lambda} \notin \Sigma_\lambda \quad \text{(defined in (4.2.13))}. \tag{4.3.20}$$

Next, we use (4.3.20) to show that

$$d_{2,\lambda} := \fint v_{2,\lambda} \text{ satisfies } \limsup_{\lambda\to+\infty} \lambda e^{d_{2,\lambda}} < +\infty, \tag{4.3.21}$$

and so by Lemma 4.2.15, we can assert that $v_{2,\lambda}$ admits the "non-topological-type" behavior (4.2.31) as claimed.

To establish (4.3.21), we argue by contradiction and assume that along a sequence $\lambda_n \to +\infty$, $\lambda_n e^{d_{2,\lambda_n}} \to +\infty$, as $n \to +\infty$. Therefore, we can apply part (2) of Lemma 4.2.15 to $v_n = v_{2,\lambda_n}$ to see that, $u_0 + v_n \to 0$ in $C^0_{\text{loc}}(\Omega \setminus \{z_1, \ldots, z_N\})$. Since $u_0 + v_{1,\lambda_c} < 0$ in $\overline{\Omega}$, for $\varepsilon > 0$ sufficiently small we find $n_\varepsilon \in \mathbb{N}: \ \forall n > n_\varepsilon$ we have:

$$v_n > v_{1,\lambda_c} \text{ in } \Omega \setminus \cup_{j=1}^N B_\varepsilon(z_j). \tag{4.3.22}$$

Setting

$$m^\varepsilon_{j,n} = \min_{||z-z_j||=\varepsilon} v_n, \ j = 1, \ldots, N,$$

for $n \geq n_\varepsilon$, we can further assume that

$$m^\varepsilon_{j,n} \geq -\left(\max_{||z-z_j||=\varepsilon} u_0 + 1\right).$$

So, if $n_j \in \mathbb{N}$ is the multiplicity of the vortex z_j, $j = 1, \ldots, N$; then

$$m^\varepsilon_{j,n} \geq 2n_j \log\frac{1}{\varepsilon} - C, \tag{4.3.23}$$

for every $n \geq n_\varepsilon$, with $C > 0$ a suitable constant independent of $\varepsilon > 0$, $n \in \mathbb{N}$ and $j \in \{1, \ldots, N\}$. We observe that

$$\begin{cases} -\Delta(v_n - m^\varepsilon_{j,n} - \pi N|z - z_j|^2) = \lambda e^{u_0+v_n}(1 - e^{u_0+v_n}) \geq 0, \text{ in } B_\varepsilon(z_j), \\ \left(v_n - m^\varepsilon_{j,n} - \pi N|z - z_j|^2\right)|_{\partial B_\varepsilon(z_j)} \geq -\pi N\varepsilon^2, \end{cases}$$

and by the maximum principle we get

$$v_n \geq m^\varepsilon_{j,n} - \pi N\varepsilon^2 \text{ in } \overline{B}_\varepsilon(z_j), \ \forall j = 1, \ldots, N.$$

Thus by virtue of (4.3.23), for $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large, we can also check that

$$v_n > v_{1,\lambda_c} \text{ in } \overline{B}_\varepsilon(z_j)\ \forall j = 1, \ldots, N. \tag{4.3.24}$$

Putting together (4.3.22) and (4.3.24), we see that $v_n > v_{1,\lambda_c}$ in Ω, and so $v_{2,\lambda_n} \in \Sigma_{\lambda_n}$, in contradiction with (4.3.20). □

Final Remarks: There has been a recent development about the "topological-type" and the "non-topological-type" solutions for (4.1.4). In [T7], the author has proved that for $\lambda > 0$ sufficiently large, the maximal solution $v_{1,\lambda}$ is the *only* solution of (4.1.4) satisfying:

$$\lambda e^{\fint_\Omega v} \to +\infty \text{ as } \lambda \to +\infty.$$

In other words, if v_n is a solution for (4.1.4) with $\lambda = \lambda_n \to +\infty$, and

$$\lambda_n e^{\fint_\Omega v_n} \to +\infty,$$

as $n \to +\infty$, then $v_n = v_{1,\lambda_n}$ for large $n \in \mathbb{N}$. This result follows from arguments similar to those presented in the proof of Theorem 3.3.14 and Theorem 3.3.15, and we refer to [T7] for details.

As a consequence, we can use Lemma 4.2.15 to conclude that for $\lambda > 0$ large, the *maximal* solution $v_{1,\lambda}$ is the *only* "topological-type" solution for (4.1.4) (in the sense of (4.1.12)). On this basis, we no longer need the (however interesting) construction above in order to obtain a "non-topological-type" solution, as we now can claim such existence directly from Theorem 4.2.14, as given by the "mountain-pass" solutions.

Recently, Choe in [Cho1] has obtained a very detailed description of the asymptotic behavior (as $\lambda \to +\infty$), of such "mountain-pass" solution. See also [ChoK] for other multiplicity results.

4.4 An alternative approach

Our next goal is to illustrate a different construction of "non-topological-type" solutions which enables us to check the convergence property in (4.2.31) with respect to the *uniform* topology.

This alternative approach was proposed in [T1] and subsequently pursued in [NT1] and [NT3]. The procedure presented below was first introduced in [CY] to handle the simpler situation of "topological-type" solutions (see also [Y1]).

To be more precise, for $v \in \mathcal{H}(\Omega)$, we use the decomposition:

$$v = w + d, \text{ with } d = \fint_\Omega v \text{ and } \int_\Omega w = 0.$$

Observe that, if v is a solution for (4.1.4), then d satisfies the equation

$$e^{2d} \int_\Omega e^{2(u_0+w)} - e^d \int_\Omega e^{u_0+w} + \frac{4\pi N}{\lambda} = 0,$$

whose solvability imposes the condition

$$\int_\Omega e^{2(u_0+w)} \le \frac{\lambda}{16\pi N}\left(\int_\Omega e^{u_0+w}\right)^2, \tag{4.4.1}$$

from which we can also re-establish the necessary condition of (4.1.8). Furthermore, when (4.4.1) holds, we can express d in terms of w as follows:

$$e^d = \frac{\int_\Omega e^{u_0+w} \pm \sqrt{\left(\int_\Omega e^{u_0+w}\right)^2 - \frac{16\pi N}{\lambda}\int_\Omega e^{2(u_0+w)}}}{2\int_\Omega e^{2(u_0+w)}}. \tag{4.4.2}$$

The two possible choice of a "plus" or "minus" sign in (4.4.2) is yet another indication of multiple existence for (4.1.4). In addition, for a solution v of (4.1.4), it is not difficult to check that

$$\text{if } \lambda e^d \to +\infty \text{ as } \lambda \to +\infty, \text{ then } d \text{ verifies (4.4.2) with the "plus" sign (for large } \lambda). \tag{4.4.3}$$

Thus, by virtue of Lemma 4.4.20, to obtain a "non-topological-type" solution, it seems reasonable to impose that (4.4.2) holds with the "minus" sign.

To this end, in the space

$$E := \left\{ w \in \mathcal{H}(\Omega) : \int_\Omega w = 0 \right\}, \tag{4.4.4}$$

we consider the subset

$$A_\lambda := \{ w \in E : \ w \text{ satisfies (4.4.1)} \}. \tag{4.4.5}$$

For $\lambda > 0$ sufficiently large, we see that $0 \in A_\lambda$, and so A_λ is not empty. Furthermore, for every $w \in A_\lambda$ we may define $d_\pm(w)$ by the property

$$e^{d_\pm(w)} = \frac{\int_\Omega e^{u_0+w} \pm \sqrt{\left(\int_\Omega e^{u_0+w}\right)^2 - \frac{16\pi N}{\lambda}\int_\Omega e^{2(u_0+w)}}}{2\int_\Omega e^{2(u_0+w)}}, \tag{4.4.6}$$

and then consider the reduced functionals

$$F_\lambda^\pm(w) = J_\lambda(w + d_\pm(w)), \ w \in A_\lambda. \tag{4.4.7}$$

Notice that $F_\lambda^\pm$ is continuous in A_λ and continuously Fréchet differentiable in $\mathring{A}_\lambda$, the interior of the set A_λ whose elements satisfy (4.4.1) with the *strict* inequality. Clearly, if $w \in \mathring{A}_\lambda$ is a critical point for $F_\lambda^\pm$, then $v = w + d_\pm(w)$ is a critical point for J_λ and hence a solution for (4.1.4). So we may search for critical points of F_λ^+ or F_λ^- in $\mathring{A}_\lambda$, in order to get solutions for (4.1.4), satisfying (4.4.2) respectively with the assigned $+$ or $-$ sign.

Towards this goal, we start to point out the following interesting property (established in [NT2]) valid for the elements of A_λ.

Lemma 4.4.20 *If $w \in A_\lambda$ and $\tau \in (0, 1]$, then*

$$\int_\Omega e^{u_0+w} \le \left(\frac{\lambda}{16\pi N}\right)^{\frac{1-\tau}{\tau}} \left(\int_\Omega e^{\tau(u_0+w)}\right)^{\frac{1}{\tau}}. \tag{4.4.8}$$

Proof. For $\tau \in (0, 1]$, let $a = \frac{1}{2-\tau} \in (0, 1)$ so that $\tau a + 2(1-a) = 1$. For $w \in A_\lambda$, we have

$$\begin{aligned}\int_\Omega e^{u_0+w} &\le \left(\int_\Omega e^{\tau(u_0+w)}\right)^a \left(\int_\Omega e^{2(u_0+w)}\right)^{1-a}\\ &\le \left(\int_\Omega e^{\tau(u_0+w)}\right)^a \left(\frac{\lambda}{16\pi N}\left(\int_\Omega e^{u_0+w}\right)^2\right)^{1-a}.\end{aligned}$$

From the inequality above and (4.4.1), the inequality (4.4.8) can be easily derived. □

On the basis of (4.4.8), we obtain the following result:

Proposition 4.4.21 *The functionals $F_\lambda^\pm$ are bounded from below and coercive in A_λ where they attain their infimum at some element $w_\lambda^\pm \in A_\lambda$. Namely,*

$$F_\lambda^\pm(w_\lambda^\pm) = \inf_{A_\lambda} F_\lambda^\pm. \tag{4.4.9}$$

Proof. As a direct consequence of (4.4.2) we see that

$$e^{d_+(w)},\ e^{d_-(w)} \ge \frac{4\pi N}{\lambda}\left(\int_\Omega e^{u_0+w}\right)^{-1},\ \forall w \in A_\lambda.$$

Therefore,

$$F_\lambda^\pm(w) = J_\lambda(w+d_\pm(w)) \ge \frac{1}{2}||\nabla w||_{L^2}^2 + 4\pi N \log\left(\frac{4\pi N}{\lambda}\right) - 4\pi N \log\left(\int_\Omega e^{u_0+w}\right),$$

and we can use (4.4.8) to deduce the estimate:

$$\begin{aligned}F_\lambda^\pm(w) &\ge \frac{1}{2}||\nabla w||_{L^2}^2 - \frac{4\pi N}{\tau}\log\left(\left(\frac{\lambda}{16\pi N}\right)^{1-\tau}\int_\Omega e^{\tau(u_0+w)}\right) + 4\pi N \log\frac{4\pi N}{\lambda}\\ &\ge \frac{1}{2}||\nabla w||_{L^2}^2 - \frac{4\pi N}{\tau}\log\int_\Omega e^{\tau w} - c_{\lambda,\tau}\end{aligned}$$

for any $\tau \in [0, 1)$ and $c_{\lambda,\tau}$ a suitable constant (depending only on λ and τ). At this point we can apply the Moser–Trudinger inequality as stated in (2.4.24), and for $0 < \tau < \min\{1, \frac{2}{N}\}$ and $\sigma = \frac{1}{2}\left(1 - \frac{N}{2}\tau\right) > 0$ derive

$$F_\lambda^\pm \ge \sigma||\nabla w||_{L^2}^2 - c_{\lambda,\tau},\ \forall w \in A_\lambda.$$

Therefore, $F_\lambda^\pm$ is bounded from below and coercive in A_λ. Hence, if $w_n^\pm$ is a minimizing sequence for $F_\lambda^\pm$ in A_λ, then $w_n^\pm$ is uniformly bounded in $\mathcal{H}(\Omega)$, and we can extract a subsequence (denoted in the same way) such that

$$w_n^\pm \to w_\lambda^\pm \text{ weakly in } \mathcal{H}(\Omega) \text{ and } e^{u_0+w_n^\pm} \to e^{u_0+w_\lambda^\pm} \text{ in } \mathrm{L}^p(\Omega),\ \forall p \geq 1.$$

In particular, $w \in A_\lambda$ and

$$d^\pm(w_n^\pm) \to d^\pm(w_\lambda^\pm).$$

Consequently,

$$\inf_{A_\lambda} F_\lambda^\pm = \lim_{n\to\infty} F_\lambda^\pm(w_n^\pm) \geq F_\lambda^\pm(w_\lambda^\pm) \geq \inf_{A_\lambda} F_\lambda^\pm,$$

and (4.4.9) is establish. □

Actually we can check that, for large λ, the minimizer w_λ^+ belongs to $\mathring{A}_\lambda$, just by comparing the minimal value that F_λ^+ attains in the boundary of A_λ:

$$\partial A_\lambda = \left\{ w \in \mathcal{H}(\Omega) : \int_\Omega w = 0 \text{ and } \int_\Omega e^{2(u_0+w)} = \frac{\lambda}{16\pi N}\left(\int_\Omega e^{u_0+w}\right)^2 \right\} \tag{4.4.10}$$

with its value in $0 \in \mathring{A}_\lambda$ (see [CY] and [Y1] for details). This furnishes an alternative proof to the existence of "topological-type" solutions for (4.1.4), when λ is sufficiently large. On the contrary, it is not as easy to determine whether or not w_λ^- belongs to $\mathring{A}_\lambda$. So far, this has been verified in [T1] and [NT3], respectively, when $N = 1$ and $N = 2$.

To illustrate the difficulty one encounters at this point, let

$$\mu = 4\pi N. \tag{4.4.11}$$

For $w \in A_\lambda$, evaluate:

$$\begin{aligned}
F_\lambda^-(w) &= J_\lambda(w + d_-(w)) = \frac{1}{2}||\nabla w||_{\mathrm{L}^2}^2 + \frac{\lambda}{2}\int_\Omega (e^{u_0+w+d_-(w)} - 1)^2 + \mu d_-(w) \\
&= \frac{1}{2}||\nabla w||_{\mathrm{L}^2}^2 + \frac{\lambda}{2}\left(1 - \frac{\mu}{\lambda}\right) - \mu\left(1 + \sqrt{1 - \frac{4\mu}{\lambda}\frac{\int_\Omega e^{2(u_0+w)}}{\left(\int_\Omega e^{u_0+w}\right)^2}}\right)^{-1} \\
&\quad + \mu\left(-\log\int_\Omega e^{u_0+w} + \log\frac{2\mu}{\lambda} - \log\left(1 + \sqrt{1 - \frac{4\mu}{\lambda}\frac{\int_\Omega e^{2(u_0+w)}}{\left(\int_\Omega e^{u_0+w}\right)^2}}\right)\right) \\
&= f_\lambda(w) + \frac{\lambda}{2} - \frac{\mu}{2} + \mu\log\frac{2\mu}{\lambda},
\end{aligned}$$

with

$$f_\lambda(w) = \frac{1}{2}||\nabla w||_{\mathrm{L}^2}^2 - \mu\log\int_\Omega e^{u_0+w} - \mu\psi\left(1 + \sqrt{1 - \frac{4\mu}{\lambda}\frac{\int_\Omega e^{2(u_0+w)}}{\left(\int_\Omega e^{u_0+w}\right)^2}}\right) \tag{4.4.12}$$

and

$$\psi(\xi) = \frac{1}{\xi} + \log \xi,\ \xi > 0. \tag{4.4.13}$$

Since the functional F_λ^- and f_λ differ only by a constant, we also see that f_λ is bounded from below in A_λ, and

$$f_\lambda(w_\lambda^-) = \inf_{A_\lambda} f_\lambda := \gamma_\lambda. \tag{4.4.14}$$

As a consequence of the fact that ψ in (4.4.13) is strictly monotone increasing for $\xi \geq 1$, we find:

Lemma 4.4.22 *The function*

$$\lambda \longrightarrow \gamma_\lambda = \inf_{A_\lambda} f_\lambda$$

is strictly monotone decreasing.

Proof. For $0 < \lambda_1 < \lambda_2$, observe that $A_{\lambda_1} \subset A_{\lambda_2}$. Therefore, for $w_j = w_{\lambda_j}^-$, $j = 1, 2$, we have

$$\begin{aligned}
\gamma_{\lambda_1} = f_{\lambda_1}(w_1) &= \frac{1}{2}||\nabla w_1||_{\mathrm{L}^2}^2 - \mu \log \int_\Omega e^{u_0 + w_1} - \mu\psi\left(1 + \sqrt{1 - \frac{4\mu}{\lambda_1}\frac{\int_\Omega e^{2(u_0+w_1)}}{\left(\int_\Omega e^{u_0+w_1}\right)^2}}\right) \\
&> \frac{1}{2}||\nabla w_1||_{\mathrm{L}^2}^2 - \mu \log \int_\Omega e^{u_0 + w_1} - \mu\psi\left(1 + \sqrt{1 - \frac{4\mu}{\lambda_2}\frac{\int_\Omega e^{2(u_0+w_1)}}{\left(\int_\Omega e^{u_0+w_1}\right)^2}}\right) \\
&= f_{\lambda_2}(w_1) \geq f_{\lambda_2}(w_2) = \gamma_{\lambda_2}.
\end{aligned}$$

□

Expression (4.4.12) brings our attention to the functional

$$I_\mu(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 - \mu \log\left(\int_\Omega e^{u_0+w}\right),\ w \in E \tag{4.4.15}$$

which we have already discussed (over a general surface) in Section 2.5 of Chapter 2.

In fact, the functional I_μ has emerged naturally from our approach, and we see that it is bounded from below, coercive in A_λ, and controls the functional f_λ from above and below as follows:

$$I_\mu(w) - \mu\psi(2) \leq f_\lambda(w) \leq I_\mu(w) - \mu\psi(1),\ \forall w \in A_\lambda.$$

More importantly, the following holds:

Lemma 4.4.23 *If*

$$f_\lambda(w_\lambda^-) = \inf_{A_\lambda} f_\lambda < \inf_{A_\lambda} I_\lambda - \mu\psi(1), \tag{4.4.16}$$

then $w_\lambda^- \in \mathring{A}_\lambda$.

Proof. Just observe that if $w_\lambda^- \in \partial A_\lambda$, then

$$f_\lambda(w_\lambda^-) = I_\mu(w_\lambda^-) - \mu\psi(1) \geq \inf_{A_\lambda} I_\mu - \mu\psi(1), \tag{4.4.17}$$

and the desired conclusion follows. □

There is a situation where (4.4.17) can be easily checked. It concernes the case $\mu \in (0, 8\pi]$, where the Moser–Trudinger inequality (2.4.24) implies that $\inf_E I_\mu > -\infty$. Hence, for every $\varepsilon > 0$, let $w_\varepsilon \in E$ be such that

$$\inf_E I_\mu \leq I_\mu(w_\varepsilon) < \inf_E I_\mu + \varepsilon.$$

Actually, for $\mu \in (0, 8\pi)$, we can simply take w_ε to coincide with the minimizer of I_μ in E, as it is always attained in this case.

For $\lambda > 0$ sufficiently large, we see that $w_\varepsilon \in A_\lambda$ and we have

$$\begin{aligned}\inf_E I_\mu - \mu\psi(2) \leq f_\lambda(w_\lambda^-) = \inf_{A_\lambda} f_\lambda \leq I_\lambda(w_\varepsilon) - \mu\psi\left(1 + \sqrt{1 - \frac{4\mu}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\varepsilon)}}{\left(\int_\Omega e^{u_0+w_\varepsilon}\right)^2}}\right)\\ \leq \inf_E I_\mu + \varepsilon - \mu\psi(2) + o(1) \text{ as } \lambda \to +\infty.\end{aligned}$$

In other words,

$$\inf_{A_\lambda} f_\lambda - \inf_E I_\mu \to -\mu\psi(2) < -\mu\psi(1), \text{ as } \lambda \to +\infty.$$

Hence, for λ large (4.4.16) holds, and we conclude the following:

Corollary 4.4.24 *For every $\mu \in (0, 8\pi]$ there exists $\lambda_\mu > 0$ such that $\forall \lambda \geq \lambda_\mu$ the extremal function w_λ^- satisfying (4.4.14) belongs in $\mathring{A}_\lambda$ and satisfies:*

$$I_\mu(w_\lambda^-) \longrightarrow \inf_E I_\mu, \tag{4.4.18}$$

$$b_\lambda := \frac{1}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\lambda^-)}}{\left(\int_\Omega e^{u_0+w_\lambda^-}\right)^2} \to 0, \text{ as } \lambda \to +\infty. \tag{4.4.19}$$

□

Thus, setting

$$w_\lambda = w_\lambda^- \text{ and } d_\lambda^- = d^-(w_\lambda^-),$$

we see that the function,

$$v_{2,\lambda} = w_\lambda + d_\lambda^-, \tag{4.4.20}$$

satisfies

$$\begin{cases} -\Delta v = \lambda e^{u_0+v}(1 - e^{u_0+v}) - \frac{\mu}{|\Omega|}, \\ v \in \mathcal{H}(\Omega). \end{cases} \tag{4.4.21}$$

In particular, if $\mu = 4\pi N$ and $N = 1, 2$, (i.e., the single and double vortex case) then $v_{2,\lambda}$ in (4.4.20) solves (4.1.4).

Since

$$\lambda e^{d_\lambda^-} = \frac{2\mu}{\int_\Omega e^{u_0+w_\lambda}\left(1+\sqrt{1-\frac{4\mu}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}}\right)}, \tag{4.4.22}$$

by (2.5.9) we see that

$$\lambda e^{d_\lambda^-} \le \frac{2\mu}{|\Omega|}. \tag{4.4.23}$$

Hence by Lemma 4.2.15 (i), we know that the solution $v_{2,\lambda}$ in (4.4.20) admits the "non-topological-type" behavior (4.2.31).

As a matter of fact, we shall see that our construction allows us to strenghten (4.2.31) as follows:

$$e^{u_0+v_{2,\lambda}} = e^{u_0+w_\lambda^-+d_\lambda^-} \to 0 \text{ as } \lambda \to +\infty, \text{ in } C(\Omega) \cap C^m(\Omega_\delta), \tag{4.4.24}$$

for any $m \in \mathbb{N}$ and $\Omega_\delta = \Omega \setminus \cup_{j=1}^N B_\delta(z_j)$, $\delta > 0$ small.

To obtain (4.4.24), we need to establish some preliminary convergence properties for w_λ. We start by observing that the property, $u_0 + v_{2,\lambda} = u_0 + w_\lambda + d_\lambda^- < 0$ in Ω, and (4.4.22) imply that

$$u_0 + w_\lambda - \log\left(\int_\Omega e^{u_0+w_\lambda}\right) \le \log\lambda - \log\mu. \tag{4.4.25}$$

In addition, by straightforward calculations we check that

$$\begin{cases} -\Delta w_\lambda = \mu\left(\frac{e^{u_0+w_\lambda}}{\int_\Omega e^{u_0+w_\lambda}} - \frac{1}{|\Omega|}\right) + f_{\lambda,\mu}, \\ w_\lambda \in \mathcal{H}(\Omega): \ \int_\Omega w_\lambda = 0 \end{cases} \tag{4.4.26}$$

with

$$f_{\lambda,\mu} = a_\lambda(\mu)\left(\frac{e^{u_0+w_\lambda}}{\int_\Omega e^{u_0+w_\lambda}} - \frac{e^{2(u_0+w_\lambda)}}{\int_\Omega e^{2(u_0+w_\lambda)}}\right), \tag{4.4.27}$$

and

$$a_\lambda(\mu) = \frac{4\mu^2}{\left(1+\sqrt{1-\frac{4\mu}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}}\right)^2}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\lambda\left(\int_\Omega e^{2(u_0+w_\lambda)}\right)^2}. \tag{4.4.28}$$

By (4.4.19), we have that $a_\lambda(\mu) = 2\mu^2 b_\lambda(1 + o(1)) \to 0$, and so $||f_{\lambda,\mu}||_{L^1(\Omega)} \to 0$ as $\lambda \to +\infty$.

We also derive

$$||f_{\lambda,\mu}\log|f_{\lambda,\mu}|||_{L^1(\Omega)} \le C_\mu b_\lambda \log\lambda, \tag{4.4.29}$$

with $C_\mu > 0$ a suitable constant independent of λ. The estimate above motivates the following:

Lemma 4.4.25 *For b_λ in (4.4.19), we have:*

$$\liminf_{\lambda\to+\infty} b_\lambda \log\lambda = 0. \tag{4.4.30}$$

Proof. We know that there exists a large $\lambda_* > 0$, such that for $\lambda \geq \lambda_*$ the function γ_λ in (4.4.14) is monotone decreasing in $[\lambda_*, +\infty)$, and therefore differentiable for a.e. $\lambda \geq \lambda_*$. Furthermore,

$$-\int_{\lambda_*}^{\lambda} \gamma_s'\,ds = \gamma_{\lambda_*} - \gamma_\lambda \longrightarrow \gamma_{\lambda_*} - \inf_E I_\mu + \mu\psi(2) := C, \text{ as } \lambda \to +\infty. \tag{4.4.31}$$

On the other hand, for $\delta > 0$ we have

$$\begin{aligned}\frac{1}{\delta}(\gamma_\lambda - \gamma_{\lambda+\delta}) &= \frac{1}{\delta}(f_\lambda(w_\lambda) - f_{\lambda+\delta}(w_{\lambda+\delta})) \geq \frac{1}{\delta}(f_\lambda(w_\lambda) - f_{\lambda+\delta}(w_\lambda))\\ &= -\frac{\mu}{\delta}\left(\psi\left(1+\sqrt{1-\frac{4\mu}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}}\right)\right.\\ &\qquad \left. -\psi\left(1+\sqrt{1-\frac{4\mu}{\lambda+\delta}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}}\right)\right),\end{aligned}$$

and passing to the limit as $\delta \to 0^+$, for a.e. $\lambda > \lambda_*$, we find:

$$\begin{aligned}-\gamma_\lambda' &\geq \mu\psi'\left(1+\sqrt{1-\frac{4\mu}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}}\right)\frac{2\mu}{\lambda^2}\frac{1}{\sqrt{1-\frac{4\mu}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}}}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}\\ &= \frac{1}{2\lambda}a_\lambda.\end{aligned}$$

Therefore, by the intergrability property of $-\gamma_\lambda'$ in $[\lambda^*, +\infty)$ (see (4.4.31)), we deduce that

$$\liminf_{\lambda\to+\infty} a_\lambda \log\lambda = 0.$$

Since $a_\lambda(\mu) = 2\mu^2 b_\lambda(1 + o(1))$ as $\lambda \to +\infty$, we conclude (4.4.30). □

By means of the blow-up analysis of the following chapter, we shall be able to replace the "lim inf" with "lim" in (4.4.30) (see Proposition 6.4.14).

We start to analyze the easier case, $\mu \in (0, 8\pi)$, where the following stronger statement holds:

Proposition 4.4.26 *If $\mu \in (0, 8\pi)$ and $m \in \mathbb{Z}^+$, then there exists a constant $C > 0$ (depending on m and μ only) such that for $v_{2,\lambda} = w_\lambda + d_\lambda^-$, we have*

$$||e^{u_0+w_\lambda}||_{C^m(\overline{\Omega})} + \lambda||e^{u_0+v_{2,\lambda}}||_{C^m(\overline{\Omega})} \leq C, \tag{4.4.32}$$

for $\lambda > 0$ sufficiently large. In particular,

$$||e^{u_0+v_{\lambda,2}}||_{C^m(\Omega)} = ||e^{u_0+w_\lambda+d_\lambda^-}||_{C^m(\overline{\Omega})} \to 0, \text{ as } \lambda \to +\infty. \tag{4.4.33}$$

Proof. Since for $\mu \in (0, 8\pi)$ the functional I_μ is coercive in E and (4.4.18) holds, then it follows that w_λ is bounded in E uniformly in λ. Consequently, the Moser–Trudinger inequality (2.4.24), applied together with Jensen's inequality (2.5.8), implies that

$$\left\|\frac{e^{2(u_0+w_\lambda)}}{\int_\Omega e^{2(u_0+w_\lambda)}}\right\|_{L^p(\Omega)} + \left\|\frac{e^{u_0+w_\lambda}}{\int_\Omega e^{u_0+w_\lambda}}\right\|_{L^p(\Omega)} \leq C, \text{ for every } p \geq 1,$$

with a constant $C > 0$ independent of λ. Hence, the right-hand side of the equation in (4.4.26) is uniformly bounded with respect to λ in $L^p(\Omega)$-norm, $\forall p \geq 1$. Thus, by elliptic estimates and a bootstrap argument, we get that w_λ is bounded in $C^m(\overline{\Omega})$, uniformly in λ, $\forall m \in \mathbb{Z}^+$. Consequently , for a suitable constant $C > 0$, we find

$$||e^{u_0+w_\lambda+d_\lambda^-(w_\lambda)}||_{C^m(\overline{\Omega})} \leq Ce^{d_\lambda^-},$$

and (4.4.32) and (4.4.33) readily follow from (4.4.23). □

From the argument above, we see that when $\mu \in (0, 8\pi)$, then along a sequence $\lambda_n \to +\infty$, $w_{\lambda_n} \to w$ in $C^m(\overline{\Omega})$, with w a minimizer for I_μ in E. Hence, w satisfies the "limiting" mean field equation of the Liouville-type:

$$\begin{cases} -\Delta w = \mu \left(\dfrac{e^{u_0+w}}{\int_\Omega e^{u_0+w}} - \dfrac{1}{|\Omega|} \right), \\ w \in \mathcal{H}(\Omega) \text{ and } \displaystyle\int_\Omega w = 0, \end{cases} \tag{4.4.34}$$

for $\mu \in (0, 8\pi)$.

At this point, it would be particularly useful to know if *uniqueness* holds for the solution of (4.4.34) or even for the minimizer of I_μ. In fact, this information would imply convergence for the whole family w_λ to such solution (or minimizer) as $\lambda \to +\infty$. However, as discussed in Section 2.5 of Chapter 2, the question of uniqueness for (4.4.34) poses a quite delicate problem, yet not resolved, in spite of the recent contributions contained in [CLS], [LiL], [LiL1], [LiW], and [LiW1]. For instance, it is natural to expect that the explicit solution of Olesen in [Ol] (obtained by a suitable use of the Weierstrass $\mathcal{P}-$function in the Liouville formula of (2.2.3)) should be the unique solution for (4.4.34) when $\mu = 4\pi$.

When $\mu = 8\pi$, it is much harder to carry out a detailed asymptotic analysis of w_λ as $\lambda \to +\infty$. Such analysis relates to the extremal property of the (no longer coercive) functional $I_{\mu=8\pi}$ as follows:

Lemma 4.4.27 *Let $\mu = 8\pi$, then $w_\lambda (= w_\lambda^-)$ in (4.4.14) is bounded in E, and uniformly so in λ, if and only if $I_{\mu=8\pi}$ attains its infimum in E.*

Proof. Clearly, if w_λ is uniformly bounded in E, then as above, we find that along a subsequence $\lambda_n \to +\infty$, w_{λ_n} converges in E (and in any other stronger $C^m(\Omega)$-norm) to a minimizer for $I_{\mu=8\pi}$. Vice versa, suppose that there exists $w_0 \in E$, such that

$$I_{\mu=8\pi}(w_0) = \inf_E I_{\mu=8\pi}.$$

Then for $\lambda > 0$ sufficiently large, we have $w_0 \in A_\lambda$ and

$$I_{\mu=8\pi}(w_0) - 8\pi\,\psi\left(1 + \sqrt{1 - \frac{32\pi}{\lambda}\frac{\int_\Omega e^{2(u_0+w_0)}}{\left(\int_\Omega e^{u_0+w_0}\right)^2}}\right) = f_\lambda(w_0)$$

$$\geq f_\lambda(w_\lambda) = I_{\mu=8\pi}(w_\lambda) - 8\pi\,\psi\left(1 + \sqrt{1 - \frac{32\pi}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}}\right)$$

$$\geq I_{\mu=8\pi}(w_0) - 8\pi\,\psi\left(1 + \sqrt{1 - \frac{32\pi}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}}\right).$$

Consequently,

$$\psi\left(1 + \sqrt{1 - \frac{32\pi}{\lambda}\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2}}\right) \geq \psi\left(1 + \sqrt{1 - \frac{32\pi}{\lambda}\frac{\int_\Omega e^{2(u_0+w_0)}}{\left(\int_\Omega e^{u_0+w_0}\right)^2}}\right)$$

and using the fact that ψ is monotone decreasing in $[1, +\infty)$, we conclude that

$$\frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2} \leq \frac{\int_\Omega e^{2(u_0+w_0)}}{\left(\int_\Omega e^{u_0+w_0}\right)^2}, \text{ for } \lambda > 0 \text{ sufficienlty large.} \tag{4.4.35}$$

Hence, recalling (4.4.26), we decompose

$$w_\lambda = w_{1,\lambda} + w_{2,\lambda}, \tag{4.4.36}$$

with $w_{1,\lambda}$ the unique solution for the problem

$$\begin{cases} -\Delta w_{1,\lambda} = 8\pi\left(\dfrac{e^{u_0+w_\lambda}}{\int_\Omega e^{u_0+w_\lambda}} - \dfrac{1}{|\Omega|}\right), \text{ in } \Omega, \\ w_{1,\lambda} \in \mathcal{H}(\Omega) \text{ and } \displaystyle\int_\Omega w_{1,\lambda} = 0, \end{cases} \tag{4.4.37}$$

and $w_{2,\lambda}$ the unique solution for the problem

$$\begin{cases} -\Delta w_{2,\lambda} = f_\lambda \text{ in } \Omega, \\ w_{2,\lambda} \in \mathcal{H}(\Omega) \text{ and } \displaystyle\int_\Omega w_{2,\lambda} = 0, \end{cases} \tag{4.4.38}$$

where f_λ is defined by (4.4.27)–(4.4.28) with $\mu = 8\pi$.

From (4.4.35), we see that the right-hand side of the equation (4.4.37) is uniformly bounded in $L^2(\Omega)$-norm, and hence

$$||w_{1,\lambda}||_{C^0(\overline{\Omega})} \le C_1, \tag{4.4.39}$$

$\forall \lambda > 0$, with a suitable constant $C_1 > 0$.

While, by Green's representation formula (cf. [Au]), we have

$$w_{2,\lambda}(z) = \frac{1}{2\pi}\int_\Omega \log\left(\frac{1}{|z-y|}\right) f_\lambda(y)\,dy + \int_\Omega \gamma(z-y) f_\lambda(y)\,dy, \tag{4.4.40}$$

with γ a smooth doubly periodic function.

Hence, by means of Jensen's inequality (2.5.6), we find

$$\begin{aligned}\int_\Omega e^{\frac{2\pi|w_{2,\lambda}|}{||f_\lambda||_{L^1(\Omega)}}}\,dx &\le \int_\Omega\int_\Omega \frac{1}{|x-y|}\frac{|f_\lambda(y)|}{||f_\lambda||_{L^1(\Omega)}}\,dxdy + C_1 \\ &\le \int_\Omega \frac{|f_\lambda(y)|}{||f_\lambda||_{L^1(\Omega)}}\left(\int_{|x-y|\le 1}\frac{1}{|x-y|}\,dxdy\right) + C_2 \le C,\end{aligned}$$

for suitable positive constants C_1, C_2, and C. Since, by (4.4.35) we know that $||f_\lambda||_{L^1(\Omega)} = O\left(\frac{1}{\lambda}\right)$, we can use the estimates above in combination with (4.4.39) to assert that

$$\forall p > 1\ \exists \lambda_p > 0 \text{ and } C_p > 0 \text{ such that } ||e^{u_0+w_\lambda}||_{L^p(\Omega)} \le C_p,\ \forall \lambda \ge \lambda_p.$$

In other words, the right-hand side of the equation in (4.4.26) is uniformly bounded in $L^p(\Omega)$ for every $p>1$, and this implies that w_λ is uniformly bounded in E as claimed. □

Lemma 4.4.27 brings to our attention the existence of a minimizer for the extremal problem:

$$\inf_E I_{\mu=8\pi}\,. \tag{4.4.41}$$

In fact, when the infimum in (4.4.41) is attained, then the conclusion of Proposition 4.4.26 holds also when we let $\mu = 8\pi$.

Remark 4.4.28 We wish to emphasize that for $\mu = 8\pi$, the family w_λ constructed above provides us with a particularly "nice" minimizing sequence for the functional $I_{\mu=8\pi}$.

Moreover, the construction presented above works equally well for the functional in (2.5.1), where we consider a more general closed surface M and assume that (2.5.3) holds.

In Section 6.3 of Chapter 6 we shall take advantage of properties (4.4.26)–(4.4.30) in order to prove that, for $M = \mathbb{R}^2/a\mathbb{Z} \times ib\mathbb{Z}$ and $a, b > 0$, the infimum of $I_{\mu=8\pi}$ is attained in E provided that there exists $p \in M : u_0(p) = \max_M u_0 > 0$, and $\Delta u_0(p) + \frac{8\pi}{|M|} > 0$.

Unfortunately such a condition is just missed if we take u_0 defined by (4.1.3) and $N = 2$. Interestingly in this case, Chen–Lin–Wang in [ChLW] have proved that the infimum in (4.4.41) is *never* attained, when in (4.1.3), we take $z_1 = z_2$ (i.e., a single vortex with double multiplicity). Even more strongly, recently in [LiW] it has been proven that the corresponding Euler–Lagrange equation (4.4.34) (with $\mu = 8\pi$) admits *no* solutions. See also [LiW1] for more recent developments in this direction.

Thus, returning to our double vortex problem, we must be ready to analyze w_λ under the circumstance that, as $\lambda \to +\infty$,

$$||\nabla w_\lambda||_{L^2(\Omega)} \to +\infty \text{ or equivalently } \left(\max_{\overline{\Omega}} w_\lambda - \log \int_\Omega e^{u_o + w_\lambda}\right) \to +\infty. \quad (4.4.42)$$

To this end, we shall need more sophisticated analytical tools involving blow-up techniques, that we will develop in the following chapter in order to conclude the following result:

Theorem 4.4.29 *Let* $\Omega := \{z = at + isb\}$ *for some* $a, b > 0$. *For* $\mu = 8\pi$, *the function* $v_{2,\lambda} = w_\lambda + d_\lambda^-$ ($w_\lambda = w_\lambda^-$ *and* $d_\lambda^- = d^-(w_\lambda^-)$) *defines a solution of (4.1.4) with* u_0 *given in (4.1.3) and* $N = 2$. *It satisfies*

$$||e^{u_0 + v_{2,\lambda}}||_{C^0(\overline{\Omega})} \to 0, \text{ as } \lambda \to +\infty. \quad (4.4.43)$$

Furthermore:
(i) Either (4.4.41) *is attained at some* $w_0 \in E$, *and along a sequence* $\lambda_n \to +\infty$, *we have*

$$\lambda_n e^{u_0 + v_{2,\lambda_n}}(1 - e^{u_0 + v_{2,\lambda_n}}) \to 8\pi \frac{e^{u_0 + w_0}}{\int_\Omega e^{u_0 + w_0}} \text{ in } L^1(\Omega); \quad (4.4.44)$$

(ii) or (4.4.41) *is not attained, and along a sequence* $\lambda_n \to +\infty$, *we have*

$$\lambda_n e^{u_0 + v_{2,\lambda_n}}(1 - e^{u_0 + v_{2,\lambda_n}}) \to 8\pi \delta_{p_0} \text{ weakly in the sense of measure in } \overline{\Omega}, \quad (4.4.45)$$

and $u_0(p_0) = \max_{\overline{\Omega}} u_0$.

Remark 4.4.30 Observe that by the non-existence of extremals for $I_{\mu=8\pi}$ derived in [ChLW], we know that alternative (ii) always holds when we consider a single vortex with double multiplicity (i.e., $z_1 = z_2$ in (4.1.3)).

To complete the proof of Theorem 4.4.29, we need to show that when (4.4.42) holds, then (4.4.43) and (4.4.45) are satisfied. This will be the goal of Section 6.4 in Chapter 6. For the moment, we wish to point out that by means of (4.4.29), we can use (4.4.40) to provide the following estimate for $w_{2,\lambda}$ in (4.4.38):

$$||\nabla w_{2,\lambda}||^2_{L^2(\Omega)} + \max_{\overline{\Omega}} |w_{2,\lambda}| = O(b_\lambda \log \lambda), \text{ as } \lambda \to +\infty.$$

Therefore, along a sequence $\lambda_n \to +\infty$ such that $b_n \log \lambda_n \to 0$ (always available by Lemma 4.4.25), for

$$w_n = w_{\lambda_n} - w_{2,\lambda_n},$$

we find

$$\begin{cases} -\Delta w_n = 8\pi \left(\frac{h_n e^{w_n}}{\int_\Omega h_n e^{w_n}} - \frac{1}{|\Omega|} \right) \text{ in } \Omega, \\ w_n \in \mathcal{H}(\Omega), \ \int_\Omega w_n = 0, \end{cases} \tag{4.4.46}$$

with

$$h_n = e^{u_0 + w_{2,n}}$$

and

$$\frac{1}{2} ||\nabla w_n||^2_{L^2(\Omega)} - 8\pi \log \left(\int_\Omega h_n e^{w_n} \right) \to \inf_E I_{\mu=8\pi}. \tag{4.4.47}$$

Notice in particular that for h_n we have

$$h_n(z) = \prod_{j=1}^{N} |z - z_j|^2 V_n(z) \text{ and } V_n \to V > 0 \text{ uniformly in } \overline{\Omega}. \tag{4.4.48}$$

We shall devote the next chapter to the asymptotic analysis of sequences for which (4.4.46) and (4.4.48) hold.

4.5 Multiple periodic Chern–Simons vortices

We can summarize the results established above in the following statement concerning periodic Chern–Simons vortices:

Theorem 4.5.31 *For a given $N \in \mathbb{N}$ and a set of points $Z := \{z_1, \ldots, z_N\} \subset \Omega$ (not necessarily distinct), there exists a constant $0 < k_0 < \sqrt{\frac{|\Omega|}{4\pi N}}$ such that the selfdual equations in (1.2.45), subject to the periodic boundary conditions of (2.1.28), admit a solution $(\mathcal{A}, \phi)_\pm$ (the $\pm$ sign chosen according to (1.2.45)) with ϕ vanishing exactly at Z (according to the given multiplicity), if and only if $k \in (0, k_0]$.*

For the solution, we have

(i) $|\phi_\pm| < \nu$ in Ω, $\phi_+(z)$ and $\overline{\phi}_-(z) = O((z - z_j)^{n_j})$ for $z \to z_j$.

(ii) The magnetic flux, electric charge and total energy are given as follows: $\Phi_\pm = \int_\Omega F_{12}^\pm = \pm 2\pi N$; $Q_\pm = \int_\Omega J_0^\pm = \pm 2\pi k N$; $E_\pm = \int_\Omega \mathcal{E}_\pm = 2\pi \nu^2 N$. Furthermore, for $k \in (0, k_0)$, there exists two classes of periodic vortex-configurations, $(\mathcal{A}^{(1)}, \phi^{(1)})_\pm$ and $(\mathcal{A}^{(2)}, \phi^{(2)})_\pm$ solutions of (1.2.45), such that:

(iii) $|\phi^{(1)}|$ is maximal among all vortex solutions whose Higgs field vanishes exactly at Z (with the given multiplicity) and, as $k \to 0^+$, it satisfies

$$\begin{aligned}
&|\phi_{\pm}^{(1)}| \to \nu \text{ in } C^m_{loc}(\Omega \setminus Z),\ \forall m \in \mathbb{Z}^+,\\
&(F_{12}^{(1)})_{\pm} \to \pm 2\pi \sum_{j=1}^{N} \delta_{z_j},\\
&(A_{\pm}^{(1)})^2 \to \pi \sum_{j\in J} n_j^2 \delta_{z_j},\\
&\tfrac{1}{k}(A_{\pm}^{(1)}) \to \pm\pi \sum_{j\in J} n_j \left(n_j+1\right) \delta_{z_j},
\end{aligned} \tag{4.5.1}$$

weakly in the sense of measure in $\overline{\Omega}$. Here $J \subset \{1, \dots, N\}$ is a set of indices identifying all distinct vortex points in Z and n_j is the multiplicity of z_j, $j \in J$.

(iv) As $k \to 0^+$,

$$|\phi_{\pm}^{(2)}| \to 0 \textit{ in } \mathrm{L}^p(\Omega),\ p \geq 1 \textit{ and pointwise a.e. in } \Omega, \tag{4.5.2}$$

and for $N = 1, 2$, the convergence in (4.5.2) holds uniformly in $\overline{\Omega}$.

The "maximal" periodic Chern–Simons vortex derived above, should be viewed as the equivalent (in the periodic setting) of the planar topological Chern–Simons vortex constructed in Chapter 3. It shares many features with the periodic Maxwell–Higgs vortex obtained in [WY], including the uniqueness property (see [T7]). However, it should be noted that the last two properties in (4.5.1) indicate that the behavior of the "maximal" Chern–Simons vortex around the vortex points is quite different from that of the Maxwell–Higgs vortex. This fact shows an important effect produced by the Chern–Simons "triple-well" potential on the corresponding vortex configurations.

To continue with our parallel to the planar case, it is natural to ask whether or not periodic vortices of the "non-topological-type" may be constructed in order to satisfy the "concentration" and the strong "localization" property (4.5.1).

Unfortunately, this is not the case for the single or double vortex solutions constructed above, which either do not concentrate at all (see (4.4.32) and (4.4.44)) or "concentrate" such that the phenomenon occurs at points different than the vortex points (see (4.4.45)).

Theorem 4.5.31 gives a rather complete description of periodic Chern–Simons vortices for the model (1.2.41) in terms of their convergence properties (for $k \to 0^+$) towards the broken asymmetric vacua states, $|\phi| = \nu$, or towards the symmetric vacuum state $\phi = 0$. In the latter case, it would be desirable to have more accurate information about the asymptotics.

The situation is not as clear for other Chern–Simons models, for example those discussed in Chapter 1.

Concerning selfdual periodic Maxwell–Chern–Simons–Higgs vortices satisfying (1.2.63) and (2.1.28), we mention that the corresponding elliptic problem (2.1.14) subject to periodic boundary conditions on $\partial\Omega$, may be reduced to a single *fourth*-order equation which admits a variational structure. Those properties have been established by Ricciardi–Tarantello in [RT1] and imply a multiplicity result analogous to that stated in Proposition 4.2.12. Furthermore, in [RT1] the authors have investigated the behavior of the MCSH-vortices in the Maxwell–Higgs limit (i.e., $\sigma \to 0,\ q$ fixed) and in the Chern–Simons limit (i.e., $\sigma \to +\infty$, but $\frac{\sigma}{q^2} = k$ fixed) and have proved that convergence holds for any reasonable norm. See also [Ri1], [Ri2], [Ri3], [ChNa], [ChN], and [ChiR] for stronger convergence results and further progress in connection with the analogous CP(1)-model.

However, results concerning the asymptotic behavior of periodic MCSH-vortices, or the validity of "concentration" properties analogous to (4.5.1) are not yet available.

Next, we discuss briefly how the approach above can be extended to treat periodic *non-abelian* $SU(n+1)$-Chern–Simons vortices, solutions of (1.3.95) (under the ansatz (1.3.116)–(1.3.117)). The corresponding system of selfdual equations (2.1.14) involve the Cartan matrix K in (2.5.17) relative to the gauge group $SU(n+1)$.

To give an idea of the type of problems one encounters in this case, while avoiding tedious technicalities, we focus on the most simple situation of physical interest, namely, that described by the gauge group $SU(3)$ (i.e., $n = 2$). Recall that the case $n = 1$ reduces to the abelian Chern–Simons model (1.2.41).

Therefore, we need to consider the 2×2 system of elliptic equations:

$$\begin{cases} \Delta u_1 = \lambda\left(4e^{2u_1} - 2e^{2u_2} - 2e^{u_1} + e^{u_2} - e^{u_1+u_2}\right) + 4\pi \sum_{j=1}^{N_1} \delta_{z_j^1}, \\ \Delta u_2 = \lambda\left(4e^{2u_2} - 2e^{2u_1} - 2e^{u_2} + e^{u_1} - e^{u_1+u_2}\right) + 4\pi \sum_{j=1}^{N_1} \delta_{z_j^2}, \end{cases} \tag{4.5.3}$$

where $\lambda = \frac{1}{k^2}$ and the points z_j^a $j = 1, \dots, N_a$ are the assigned zeroes (not necessarily distinct) of the complex field ϕ^a, $a = 1, 2$, involved in the decomposition (1.3.117). Also recall that: $|\phi^a|^2 = e^{u_a}$, with a zero set denoted by $Z_a = \left\{z_1^a, \dots z_{N_a}^a\right\}$, $a = 1, 2$.

The $SU(3)$ potential V (restricted according to (1.3.102)) takes the following form:

$$V = \frac{1}{4k^2}\left(|\phi^1|^2\left(2|\phi^1|^2 - |\phi^2|^2 - 1\right)^2 + |\phi^2|\left(2|\phi^2|^2 - |\phi^1|^2 - 1\right)^2\right). \tag{4.5.4}$$

The zeroes of V give the vacua states of the system and are specified as follows:

(i) broken principal embedding vacua states, $|\phi^1| = |\phi^2| = 1$;
(ii) symmetric unbroken vacuum state, $\phi^1 = \phi^2 = 0$;
(iii) "mixed" vacua states (absent in the abelian situation), $|\phi^1|^2 = \frac{1}{2}$, $\phi^2 = 0$ or $\phi^1 = 0$, $|\phi^2|^2 = \frac{1}{2}$.

Therefore in this case, the nature of the corresponding vortex configurations should be characterized according to *three* different asymptotic behaviors, as $k \to 0^+$, namely:

type I (topological type): $a = 1, 2$

$$|\phi^a| \to 1; \tag{4.5.5}$$

type II (non-topological type): $a = 1, 2$

$$|\phi^a| \to 0; \tag{4.5.6}$$

type III (mixed topological / non-topological type):

$$|\phi^1|^2 \to \frac{1}{2},\ |\phi^2| \to 0 \text{ or } |\phi^1| \to 0,\ |\phi^2|^2 \to \frac{1}{2}. \tag{4.5.7}$$

Clearly, the presence of the symmetric and the principal embedding vacua states persist for the more general $SU(n+1)$-Chern–Simons model, while the mixed vacua states are going to vary with $n \in \mathbb{N}$, and give rise to various other types of possible asymptotic behaviors. For the *planar* case, the existence of a topological $SU(n+1)$-vortex (where (4.5.5) holds as $|z| \to +\infty$ for every $a = 1, \ldots, n$) has been established by Yang in [Y6] by means of an iteration scheme, in the same spirit of the abelian case (see [Y1]). Concerning the non-topological case (where (4.5.6) holds as $|z| \to +\infty$ and $a = 1, \ldots, n$,), as well as, the other "mixed" cases, the question of existence is completely open (even for the case $n = 2$), except for an attempt in [WZ] by Wang and Yang, whose proof, however, presents a gap.

For the *periodic* case, the existence of Chern–Simons $SU(3)$-vortices has been handled by Nolasco–Tarantello in [NT1], where the authors are able to treat all three types of asymptotic behaviors of (4.5.5), (4.5.6), and (4.5.7) by introducing constraint variational problems inspired by those discussed above for the abelian setting. More precisely, for $a = 1, 2$, we introduce the function u_0^a as the unique solution for the problem

$$\begin{cases} \Delta u_0^a = 4\pi \sum_{j=1}^{N_a} \delta_{z_j^a} - \dfrac{4\pi N_a}{|\Omega|} \text{ in } \Omega, \\ u_0^a \text{ doubly periodic on } \partial\Omega, \end{cases} \tag{4.5.8}$$

(see [Au]) and set

$$u_a = u_0^a + v_a, \tag{4.5.9}$$

$$h_a = e^{u_0^a}. \tag{4.5.10}$$

Then, to obtain periodic $SU(3)$-Chern–Simons vortices, we need to find pairs (v_1, v_2) that solve the problem:

$$\begin{cases} \Delta v_1 = \lambda(4h_1^2 e^{2v_1} - 2h_2^2 e^{2v_2} - 2h_1 e^{v_1} + h_2 e^{v_2} - h_1 h_2 e^{v_1+v_2}) + \frac{4\pi N_1}{|\Omega|}, \text{ in } \Omega, \\ \Delta v_2 = \lambda(4h_2^2 e^{2v_2} - 2h_1^2 e^{2v_1} - 2h_2 e^{v_2} + h_1 e^{v_1} - h_1 h_2 e^{v_1+v_2}) + \frac{4\pi N_2}{|\Omega|}, \text{ in } \Omega, \\ v_1, v_2 \text{ doubly periodic on } \partial\Omega. \end{cases} \tag{4.5.11}$$

As for the abelian case, the solution of (4.5.11) satisfies:

$$e^{u_a} = e^{u_a^0 + v_a} < 1 \text{ in } \Omega;\ a = 1, 2. \tag{4.5.12}$$

Indeed, notice that u_a attains its maximum value in $\overline{\Omega}$ at a point $z^a \in \overline{\Omega} \setminus Z_a$ where $\Delta u_a(z^a) \leq 0$. Set $u_a(z^a) = \max_{\overline{\Omega}} u_a = \overline{u}_a$. By symmetry we can assume (without loss of generality) that $\overline{u}_1 \geq \overline{u}_2$. Therefore, from (4.5.3) we find

$$\begin{aligned} 0 \geq \Delta u_1(z^1) &= \lambda \left(4e^{2\overline{u}_1} - 2e^{2u_2(z^1)} - 2e^{\overline{u}_1} + e^{u_2(z^1)} - e^{\overline{u}_1 + u_2(z^1)}\right) \\ &\geq \lambda \left(2e^{2\overline{u}_1} - 2e^{\overline{u}_1} + e^{u_2(z^1)} - e^{\overline{u}_1 + u_2(z^1)}\right) = \lambda(2e^{\overline{u}_1} - e^{u_2(z^1)}) \left(e^{\overline{u}_1} - 1\right), \end{aligned}$$

and conclude that necessarily $e^{\overline{u}_1} < 1$ in Ω and (4.5.12) follows.

In addition, it is easy to check that (weak) solutions for (4.5.11) correspond to critical points in $\mathcal{H}(\Omega) \times \mathcal{H}(\Omega)$ of the functional

$$\begin{aligned} J_\lambda(v_1, v_2) = &\frac{1}{3}\left(\int_\Omega |\nabla v_1|^2 + |\nabla v_2|^2 + \nabla v_1 \nabla v_2\right) + \lambda \int_\Omega W(v_1, v_2) \\ &+ \frac{4\pi}{3}(2N_1 + N_2) \fint_\Omega v_1 + \frac{4\pi}{3}(2N_2 + N_1) \fint_\Omega v_2, \end{aligned} \tag{4.5.13}$$

where

$$\begin{aligned} W(v_1, v_2) &= h_1^2 e^{2v_1} + h_2^2 e^{2v_2} - h_1 e^{v_1} - h_2 e^{v_2} - h_1 h_2 e^{v_1 + v_2} + 1 &(4.5.14)\\ &= \frac{1}{4}\left(2h_1 e^{v_1} - h_2 e^{v_2} - 1\right)^2 + \frac{3}{4}\left(h_2 e^{v_2} - 1\right)^2 &(4.5.15)\\ &= \frac{1}{4}\left(2h_2 e^{v_2} - h_1 e^{v_1} - 1\right)^2 + \frac{3}{4}\left(h_1 e^{v_1} - 1\right)^2 \geq 0. &(4.5.16) \end{aligned}$$

On the other hand, if for a solution pair (v_1, v_2), we use the decomposition

$$v_a = w_a + d_a, \text{ with } d_a = \fint_\Omega v_a \text{ and } \int_\Omega w_a = 0, \ a = 1, 2;$$

then, by integrating the equations in (4.5.11) over Ω, we obtain the following quadratic equations for e^{d_1} and e^{d_2}:

$$\begin{cases} 2e^{2d_1} \int_\Omega h_1^2 e^{2w_1} - e^{d_1}\left(\int_\Omega h_1 e^{w_1} + e^{d_2} \int_\Omega h_1 h_2 e^{w_1 + w_2}\right) + \frac{4\pi}{3\lambda}(2N_1 + N_2) = 0, \\ 2e^{2d_2} \int_\Omega h_2^2 e^{2w_2} - e^{d_2}\left(\int_\Omega h_2 e^{w_2} + e^{d_1} \int_\Omega h_1 h_2 e^{w_1 + w_2}\right) + \frac{4\pi}{3\lambda}(2N_2 + N_1) = 0. \end{cases} \tag{4.5.17}$$

The constraints (4.5.17) show that necessarily:

$$\frac{\left(\int_\Omega h_a e^{w_a} + e^{d_b} \int_\Omega h_a h_b e^{w_a + w_b}\right)^2}{\int_\Omega h_a^2 e^{2w_a}} \geq \frac{32\pi(2N_a + N_b)}{3\lambda}, \ a \neq b \in \{1, 2\}.$$

By (4.4.45) we also know that $h_b e^{w_b + d_b} < 1$ in Ω, and we derive the estimate:

$$\frac{32\pi(2N_a + N_b)}{3\lambda} \leq \frac{4\left(\int_\Omega h_a e^{w_a}\right)^2}{\int_\Omega h_a^2 e^{2w_a}} \leq 4|\Omega|, \ a \neq b \in \{1, 2\}.$$

Hence, for the solvability of (4.5.11), it is necessary that:

$$\lambda > \frac{8\pi}{3|\Omega|} \max\{2N_1 + N_2, 2N_2 + N_1\}. \tag{4.5.18}$$

More importantly, in [NT1] it is shown that for a pair $(w_1, w_2) \in E \times E$ satisfying

$$\frac{\int_\Omega h_a^2 e^{2w_a}}{\left(\int_\Omega h_a e^{w_a}\right)^2} \le \frac{3\lambda}{32\pi(2N_a + N_b)}, \quad a \ne b \in \{1, 2\}, \tag{4.5.19}$$

it is possible to obtain *four* pairs of solutions for (4.5.17), that depend on w_1 and w_2 and that are specified according to the possible choices of the "+" or "−" sign in the following equivalent formulation of the system (4.5.17):

$$\begin{aligned} e^{d_a} = {} & \frac{\int_\Omega h_a e^{w_a} + e^{d_b} \int_\Omega h_1 h_2 e^{w_1+w_2}}{4\int_\Omega h_a^2 e^{2w_a}} \\ & \pm \frac{\sqrt{\left(\int_\Omega h_a e^{w_a} + e^{d_b}\int_\Omega h_1 h_2 e^{w_1+w_2}\right)^2 - \frac{32\pi}{3\lambda}(2N_a + N_b)\int_\Omega h_a^2 e^{2w_a}}}{4\int_\Omega h_a^2 e^{2w_a}}, \end{aligned} \tag{4.5.20}$$

where $a \ne b \in \{1, 2\}$.

Thus, we denote with (d_1^+, d_2^+) and (d_1^-, d_2^-) the solutions for (4.5.20) where we pick the "+" sign or the "−" sign, respectively, in both equations in (4.5.20). Here $(d_1^\pm, d_2^\mp)$ (respectively $(d_1^\mp, d_2^\mp)$) denotes the solution for which we pick: the "+" sign (respectively, the "−" sign), in the equation corresponding to $a = 1$ in (4.5.20); and the "−" sign (respectively, the "+" sign), in the equation corresponding to $a = 2$ in (4.5.20). We refer to [NT1] for the detailed proof on the existence of such solutions. Therefore, as for the abelian case, on the set

$$A_\lambda = \{(w_1, w_2) \in E \times E : (4.5.18) \text{ holds }\} \subset E \times E, \tag{4.5.21}$$

we define *four* constraint functionals:

$$F_\lambda^*(w_1, w_2) = J_\lambda(w_1 + d_1^*, w_2 + d_2^*) \text{ with } * = + \text{ or } - \tag{4.5.22}$$

and

$$F_\lambda^\#(w_1, w_2) = J_\lambda(w_1 + d_1^\#, w_2 + d_2^\#) \text{ with } \# = \pm \text{ or } \mp. \tag{4.5.23}$$

Since (4.5.18) has the same structure as (4.4.1) (in fact, reduces to it when $N_1 = N_2$), it is not difficult to check that a result similar to Lemma 4.4.20 remains valid for $e^{u_a^0 + w_a} = h_a e^{w_a}$, $a = 1, 2$. Furthermore, in terms of the $K^{-1} = (K^{ij})_{i,j=1,2}$, the inverse of the Cartan 2×2 matrix, $K = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, we have

$$\begin{aligned} J_\lambda(w_1 + d_1, w_2 + d_2) = {} & \frac{1}{2}\sum_{i,j=1}^{2} K^{ij} \nabla w_i \cdot \nabla w_j + \lambda \int_\Omega W(w_1 + d_1, w_2 + d_2) \\ & + \frac{4\pi}{3}\big((2N_1 + N_2)d_1 + (2N_2 + N_1)d_2\big), \end{aligned} \tag{4.5.24}$$

with W given in (4.5.14). As in Proposition 4.4.21 the functionals in (4.5.22) and (4.5.23) are bounded from below and coercive in A_λ, where they attain their infimum. See [NT1] for details.

Hence, we let $\left(w^*_{1,\lambda}, w^*_{2,\lambda}\right) \in A_\lambda$

$$F^*_\lambda\left(w^*_{1,\lambda}, w^*_{2,\lambda}\right) = \inf_{A_\lambda} F^*_\lambda, \tag{4.5.25}$$

with $* = +$ or $-$;

and for $\left(w^\#_{1,\lambda}, w^\#_{2,\lambda}\right) \in A_\lambda$,

$$F^\#_\lambda\left(w^\#_{1,\lambda}, w^\#_{2,\lambda}\right) = \inf_{A_\lambda} F^\#_\lambda, \tag{4.5.26}$$

with $\# = \pm$ or $\mp$.

We need to check that these extremal functions belong to the *interior* of A_λ, in order to yield a solution for (4.5.11) as given by:

$$v^*_{a,\lambda} = w^*_{a,\lambda} + d^*_a,\ a = 1, 2, \tag{4.5.27}$$

with $* = +$ or $-$; and

$$v^\#_{a,\lambda} = w^\#_{a,\lambda} + d^\#_a,\ a = 1, 2, \tag{4.5.28}$$

with $\# = \pm$ or $\mp$.

For $* = +$, we expect (4.5.27) to satisfy the *topological* asymptotic behavior

$$e^{u^0_a + v^+_{a,\lambda}} \to 1,\ a = 1, 2;\ \text{as } \lambda \to +\infty. \tag{4.5.29}$$

While for $* = -$, we expect (4.5.28) to satisfy the *non-topological* asymptotic behavior

$$e^{u^0_a + v^-_{a,\lambda}} \to 0,\ a = 1, 2;\ \text{as } \lambda \to +\infty. \tag{4.5.30}$$

Similarly, for $\# = \pm$ or $\# = \mp$, we expect (4.5.28) to satisfy respectively one of the following *mixed topological/non-topological* condition

$$e^{u^0_1 + v^\pm_{1,\lambda}} \to \frac{1}{2} \text{ and } e^{u^0_2 + v^\pm_{2,\lambda}} \to 0, \tag{4.5.31}$$

or

$$e^{u^0_1 + v^\mp_{1,\lambda}} \to 0 \text{ and } e^{u^0_2 + v^\mp_{2,\lambda}} \to \frac{1}{2}; \tag{4.5.32}$$

as $\lambda \to +\infty$.

As for the abelian case, this task can be carried out more easily for the topological case, corresponding to the choice $* = +$, and leads to the following result:

Theorem 4.5.32 *[NT1] For sufficiently large λ, we have:*

$$\inf_{\partial A_\lambda} F_\lambda^+ > \min_{A_\lambda} F_\lambda^+ = F_\lambda^+\left(w_{1,\lambda}^+, w_{2,\lambda}^+\right);$$

and therefore $(v_{a,\lambda}^+ = w_{a,\lambda}^+ + d_a^+)_{a=1,2}$ defines a solution for (4.5.11) and satisfies (4.5.29) in $L^p(\Omega)$, $p \geq 1$, and pointwise a.e. in Ω. Actually, $(v_{1,\lambda}^+, v_{2,\lambda}^+)$ defines a local minimum for the functional J_λ in (4.5.13).

Furthermore J_λ admits also another critical point in $E \times E$.

Thus, as for the abelian case, we can always ensure multiple solutions for (4.5.11), and also guarantee that one such solution always admits a *topological* behavior, in the sense that (4.5.29) holds, as $\lambda \to +\infty$. We refer to [NT1] for details.

On the contrary, the other cases are more delicate to handle and require further restrictions on the vortex numbers $N_a,\ a = 1, 2$.

Concerning the choice of $* = -$, which we expect to yield the non-topological behavior (4.5.29), the condition,

$$\inf_{\partial A_\lambda} F_\lambda^- > \min_{A_\lambda} F_\lambda^- = F_\lambda^-\left(w_{1,\lambda}^-, w_{2,\lambda}^-\right), \tag{4.5.33}$$

was checked to hold by Nolasco–Tarantello in [NT1] under the assumption that $N_1 + N_2 = 1$. In this situation, the minimizer $(w_{1,\lambda}^-, w_{2,\lambda}^-)$ also satisfies a strong convergence property (along a sequence $\lambda_n \to +\infty$) towards a minimizer of the Toda functional:

$$I_{\mu_1,\mu_2}(w_1, w_2) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{2} K^{ij}\nabla w_i \cdot \nabla w_j - \sum_{j=1}^{2} \mu_i \log\left(\int_\Omega h_j e^{w_j}\right),\ w_1, w_2 \in E, \tag{4.5.34}$$

with

$$\mu_1 = \frac{4\pi}{3}(2N_1 + N_2) \text{ and } \mu_2 = \frac{4\pi}{3}(2N_2 + N_1). \tag{4.5.35}$$

This can be explained by recalling expression (4.5.24) for J_λ and that according to Jost–Wang's result in [JoW1] (see Theorem 1.2.13), the functional in (4.5.34) is bounded from below if and only if $\mu_j \leq 4\pi$. Moreover, the infimum is always attained when the strict inequality holds, i.e., when $\mu_j < 4\pi,\ \forall j = 1, 2$. Clearly if $N_1 + N_2 = 1$, then μ_1 and μ_2 in (4.5.35) are both less then 4π; this justifies the need for such restriction on the vortex numbers N_1 and N_2. On the other hand, the functional in (4.5.34) remains bounded from below also when μ_1 and μ_2 are given by (4.5.35) with $N_1 = N_2 = 1$. In this case, $\mu_1 = \mu_2 = 4\pi$ and we are at a borderline situation, analogous to that encountered for the double vortex case (i.e., N $= 2$) in the abelian Chern–Simons 6th-order model.

Indeed, one can proceed in an analogous way (see [NT1]) to arrive at the following conclusion:

Theorem 4.5.33 *([NT1]) If $N_a \in \{0,1\}$, $a = 1,2$ and $\lambda > 0$ is sufficiently large, then the minimizer $(w^-_{1,\lambda}, w^-_{2,\lambda})$ of F^-_λ belongs to the interior of the set A_λ in (4.5.21). Hence $v^-_{a,\lambda} = (w^-_{a,\lambda} + d^-_a)_{a=1,2}$ defines a solution for (4.5.11) that admits the following "non-topological-type" asymptotic behavior,*

$$e^{u^0_a + v^-_{a,\lambda}} \to 0 \text{ uniformly in } \overline{\Omega} \text{ as } \lambda \to +\infty, \quad a = 1,2. \tag{4.5.36}$$

For details we refer to [NT1] for the case $N_1 + N_2 = 1$, for which the convergence in (4.5.36) is actually shown to hold with respect to $C^m(\overline{\Omega})$-norm. In addition, along a sequence $\lambda_n \to +\infty$, it is shown that $w^-_{a,\lambda_n} \to w_a$ in E with $a = 1,2$, and (w_1, w_2) a solution of the 2×2 Toda system in mean field form (recall (2.5.25)):

$$\begin{cases} -\Delta w_1 = \dfrac{4\pi}{3}\left((2N_1 + N_2)\dfrac{2h_1 e^{w_1}}{\int_\Omega h_1 e^{w_1}} - (2N_2 + N_1)\dfrac{h_2 e^{w_2}}{\int_\Omega h_2 e^{w_2}}\right) - \dfrac{4\pi N_1}{|\Omega|,} \text{ in } \Omega, \\ -\Delta w_2 = \dfrac{4\pi}{3}\left((2N_2 + N_1)\dfrac{2h_2 e^{w_2}}{\int_\Omega h_2 e^{w_2}} - (2N_1 + N_2)\dfrac{h_1 e^{w_1}}{\int_\Omega h_1 e^{w_1}}\right) - \dfrac{4\pi N_2}{|\Omega|} \text{ in } \Omega, \\ w_1, w_2 \in \mathcal{H}(\Omega), \ \int_\Omega w_1 = 0 = \int_\Omega w_2. \end{cases} \tag{4.5.37}$$

More precisely, we know that (w_1, w_2) defines a minimum for the functional in (4.5.34) and in (4.5.35).

The more delicate case, $N_1 = N_2 = 1$, follows by arguments similar to those given above for the abelian Chern–Simons 2-vortex problem, in combination with the results in [JoW1] and [JoW2]. Analogously, in this situation, one can no longer guarantee strong convergence towards a solution of the system (4.5.37). In fact, for $N_1 = N_2 = 1$, the existence of a solution for (4.5.37) becomes a delicate issue that in general cannot be dealt by using a simple minimization procedure.

A possibility would be to suppose that h_1 and h_2 satisfy conditions analogous to (2.5.4) (as shown in [JoLW] to which we refer for details); however this assumption, does not allow us to treat our choice of h_a in (4.5.8) and (4.5.10) for $a = 1,2$.

Finally, concerning the minimization problem in (4.5.26), the condition $N_1 + N_2 = 1$ is still sufficient to guarantee that either one of the minimizers, $(w^\pm_{1,\lambda}, w^\pm_{2,\lambda})$ for $F^\pm_\lambda$ or $(w^\mp_{1,\lambda}, w^\mp_{2,\lambda})$ for $F^\mp_\lambda$, belongs to the interior of the set A_λ (in (4.5.21)), when λ is sufficiently large. More precisely the following holds:

Theorem 4.5.34 *([NT1]) If $N_1 = 0$ and $N_2 = 1$, then for $\lambda > 0$ sufficiently large the minimizer $(w^\pm_{1,\lambda}, w^\pm_{2,\lambda})$ of $F^\pm_\lambda$ belongs to the interior of A_λ, so that $(v^\pm_{a,\lambda}) = (w^\pm_{a,\lambda} + d^\pm_{a,\lambda})_{a=1,2}$ defines a solution for (4.5.11). Furthermore,*

$$w^\pm_{1,\lambda} \to 0, \text{ as } \lambda \to +\infty, \text{ and } w^\pm_{2,\lambda} \text{ is uniformly bounded in } E;$$
$$d^\pm_1 \to \log\frac{1}{2} \text{ and } d^\pm_2 \to -\infty \text{ as } \lambda \to +\infty$$

In particular,

$$e^{v^\pm_{1,\lambda}} \to \frac{1}{2} \text{ and } e^{u^0_2 + v^\pm_{2,\lambda}} \to 0, \text{ as } \lambda \to +\infty.$$

(Notice that if $N_1 = 0$, then $u^0_1 \equiv 0$.)

If $N_1 = 1$ and $N_2 = 0$, then the conclusion above holds for the minimizer $(w^{\mp}_{1,\lambda}, w^{\mp}_{2,\lambda})$ of $F^{\mp}_{\lambda}$, with the role of the index $a = 1$ and $a = 2$ exchanged.

If $N_1 = 0$ and $N_2 = 1$, (or if $N_1 = 1$ and $N_2 = 0$), then $w^{\pm}_{2,\lambda}$ (or $w^{\mp}_{1,\lambda}$) is uniformly bounded in E, and along a sequence $\lambda_n \to +\infty$, we find $w^{\pm}_{2,\lambda_n}$ (or $w^{\mp}_{2,\lambda_n}$) converges (weakly) in E to a solution of the mean field equation (4.4.34) with $\mu = 4\pi$.

In terms of periodic Chern–Simons $SU(3)$-vortices, the results above can be summarize as follows:

Theorem 4.5.35 *For $a = 1, 2$, let $N_a \in \mathbb{Z}^+$ and $Z_a = \left\{z^a_1, \ldots, z^a_{N_a}\right\} \subset \Omega$ be a given set of points repeated according to their multiplicity. There exists $0 < k_* < \sqrt{\frac{3|\Omega|}{8\pi \max\{2N_1+N_2, 2N_2+N_1\}}}$, such that for every $k \in (0, k^*)$, we have two distinct periodic Chern–Simons $SU(3)$-vortices solutions for (1.3.115) under the ansatz (1.4.26), (1.4.27) and the boundary conditions (2.1.34). Furthermore,*

(i) The component ϕ^a of the Higgs field satisfies $|\phi^a| < 1$ in Ω, and ϕ^a vanishes exactly at each point of Z_a with the given multiplicity, for $a = 1, 2$.
The component Φ_a of the magnetic field and the component Q_a of the electric charge, respectively, verify

$$\Phi_a = \frac{1}{k} Q_a = \frac{2\pi}{3}(2N_a + N_b), \; a \neq b \in \{1, 2\};$$

and for the energy E, we have

$$E = 2\pi(N_1 + N_2).$$

(ii) One of the given solutions is always of the "topological-type", in the sense that (4.5.2) holds in $L^p(\Omega)$ $p \geq 1$ and pointwise a.e. in Ω, as $k \to 0^+$.
(iii) If $N_a \in \{0, 1\}$ $a = 1, 2$, then there exists a solution of the "non-topological-type" which satisfies (4.5.6) uniformly in $\overline{\Omega}$, as $k \to 0^+$.
(iv) If $N_1 + N_2 = 1$, then there also exists a solution of the "mixed" topological-/non-topological-type for which one of the asymptotic behaviors in (4.5.7) holds in $\mathcal{H}(\Omega)$, as $k \to 0$.
(v) For $k > \sqrt{\frac{3|\Omega|}{8\pi \max\{2N_1+N_2, 2N_2+N_1\}}}$, NO such solutions exists.

As for the abelian case, it is reasonable to expect that conclusions (iii) and (iv) of Theorem 4.5.35 should hold without any restriction on the vortex numbers N_a, $a = 1, 2$.

4.6 Final remarks and open problems

Again we conclude our discussion of periodic Chern–Simons vortices by pointing out some of the related open problems.

Firstly, the discussion above explains and re-enforces our interest in the existence or non-existence of extremals for the Moser–Trudinger inequality, a question to which we provide a partial answer in the following chapter.

This question now holds even more relevance in the context of systems, and more specifically, for the $SU(3)$-Toda functional (4.5.34), where we ask:

Open question: Does the functional (4.5.34) with $N_a = 1$ and h_a defined in (4.5.8) and (4.5.10) with $a = 1, 2$, attain its infimum in $E \times E$?

Furthermore, how does the answer to this question depend on the shape of the periodic cell domain Ω (i.e., rectangular, rhombus, etc.)?

Also, for a rectangular domain Ω, does the condition that the points z_1^a, $a = 1, 2$ (in 4.5.8), coincide or not affect the answer?

In this respect, recall the results in [LiW] for the single equation.

Clearly, the existence of extremals can be more generally asked for the $SU(n+1)$-Toda functional over a closed surface. As usual, we expect the case of the standard sphere $M = S^2$ to contain many elements of interest. We mention that the only available result in this context is contained in [JoLW].

An answer to the above questions would permit us to construct non-abelian Chern–Simons vortices satisfying appropriate "concentration" and strong "localization" properties around some points, in a manner consistent with what is observed in the physical applications.

As a matter of fact, even in the abelian case and for any vortex number $N \geq 2$, it is not known whether periodic Chern–Simons vortices exist which asymptotically satisfy the "non-topological-type" condition (4.5.2), and at the same time, admit a "concentration" behavior similar to that in (4.5.1).

Any progress in this direction would certainly give indications on how to carry out similar constructions also for $SU(3)$-vortices, possibly with respect to both "non-topological," and "mixed-type" asymptotic behaviors, as given in (4.5.6) and (4.5.7), respectively.

In any event, removing the restriction on the vortex numbers N_a, $a = 1, 2$, from Theorem 4.3.35 would serve as an important, step toward the understanding of $SU(3)$-vortices. This amounts to provide an affirmative answer to the following:

Open question: Consider any given $N_a \in \mathbb{N}$ and h_a satisfying (4.5.8), (4.5.10), $a = 1, 2$. Does problem (4.5.11) admit a solution $(v_1, v_2)_\lambda$ for large $\lambda > 0$ such that for $\lambda \to +\infty$,

$$h_a e^{v_a} \to 0 \quad \text{a.e. in } \Omega, a = 1, 2$$

(non-topological-type); and a second solution $(\tilde{v}_1, \tilde{v}_2)_\lambda$, for $\lambda > 0$ large, such that as $\lambda \to +\infty$,

$$h_1 e^{v_1} \to \frac{1}{2} \text{ and } h_2 e^{v_2} \to 0 \quad \text{or} \quad h_1 e^{v_1} \to 0 \text{ and } h_2 e^{v_2} \to \frac{1}{2}$$

a.e in Ω (mixed-type)?

5

The Analysis of Liouville-Type Equations With Singular Sources

5.1 Introduction

The construction of periodic "non-topological-type" Chern–Simons vortices developed in Section 4.4 of Chapter 4 has lead us to consider sequences w_k defined over the (periodic cell) domain Ω satisfying

$$\begin{cases} -\Delta w_k = \mu_k \left(\dfrac{h_k e^{w_k}}{\int_\Omega h_k e^{w_k}} - \dfrac{1}{|\Omega|} \right) \text{ in } \Omega, \\ w_k \in \mathcal{H}(\Omega) : \displaystyle\int_\Omega w_k = 0, \end{cases} \tag{5.1.1}$$

where $\mu_k > 0$ is a given bounded sequence and the weight function h_k takes the form

$$h_k(z) = \prod_{j=1}^{m} |z - p_j|^{2\alpha_j} V_k(z) \text{ and } 0 < b_1 \leq V_k \leq b_2 \text{ in } \overline{\Omega}, \tag{5.1.2}$$

with $p_1, \ldots, p_m \in \overline{\Omega}$ the *distinct* zeroes of h_k (corresponding to the distinct set of vortex points) and relative multiplicity $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^+$ (see (4.4.46) and (4.4.47)).

More precisely, we are interested in analyzing the asymptotic behavior of w_k in the situation where

$$\max_{\overline{\Omega}} w_k - \log \int_\Omega h_k e^{w_k} \to +\infty, \text{ as } k \to +\infty \quad \text{(see (4.4.42))}. \tag{5.1.3}$$

By the interpretation of such solutions as vortex configurations, we expect them to "concentrate" at certain points that are likely to coincide with the vortex points. For this reason, we start by localizing our analysis around a vortex point and by denoting $\alpha > 0$ as the corresponding multiplicity. After a translation, we can always take the vortex point to coincide with the origin. Moreover, if we work with the new sequence

$$u_k = w_k - \log \left(\int_\Omega h_k e^{w_k} \right),$$

then we are lead to study the following "local" problem:

$$\begin{cases} -\Delta u_k = |z|^{2\alpha} V_k(z) e^{u_k} \text{ in } B_\delta(0) = \{|z| < \delta\}, \\ \int_{B_\delta(0)} |z|^{2\alpha} V_k(z) e^{u_k} \leq C, \end{cases} \tag{5.1.4}$$

where $\delta > 0$ small, $C > 0$ is a suitable constant, and $0 < b_1 \leq V_k \leq b_2$ in $B_\delta(0)$. To give an unified discussion, we shall take $\alpha \geq 0$, with the understanding that $\alpha = 0$ describes the situation where we "localize" our analysis away from the vortex points.

Our first task will be to establish a "concentration-compactness" alternative for (a subsequence of) the sequence $|z|^{2\alpha} e^{u_k}$ (cf. [Lns]). In the case of concentration, (i.e., when (5.1.3) holds), we shall use blow-up techniques in order to describe rather accurately the "bubbling" behavior of u_k.

For instance, we shall see that, if u_k admits uniformly bounded oscillations on $\partial B_\delta(0)$, namely,

$$\sup_{\partial B_\delta(0)} u_k - \inf_{\partial B_\delta(0)} u_k \leq C, \tag{5.1.5}$$

for a suitable constant $C > 0$, then the formation of "multiple bubbles" is not possible, while such phenomenon is known to occur when (5.1.5) is violated (see examples in Section 5.5.5).

Notice that condition (5.1.5) can always be checked for our original sequence w_k in (5.1.1).

In the "single-bubble" situation, we shall be able to provide pointwise estimates on the profile of the sequence u_k around its "bubbling point". This analysis is based upon an appropriate "inf+sup" estimate for solutions of the equation

$$-\Delta u = |z|^{2\alpha} V(z) e^u \text{ in } B_1(0), \tag{5.1.6}$$

which holds independent interest. More precisely, it can be shown that, if $\alpha \geq 0$ and $V \in C^{0,1}(B_1)$ satisfies $0 < b_1 \leq V \leq b_2$, $|\nabla V| \leq A$ in B_1, then every solution of (5.1.6) satisfies

$$u(0) + \inf_{B_1} u \leq C, \tag{5.1.7}$$

with a universal constant $C > 0$ depending only α, b_1, b_2, and A.

After our "local" analysis is completed, we shall patch together all such "local" information to obtain a "global" concentration/compactness principle for a solution-sequence satisfying (5.1.1), (5.1.2) (see Theorem 5.4.34 and Theorem 5.7.61 below). By such results in Chapter 6, we shall be able to complete the proof of Theorem 4.4.29 and also obtain an existence result for a generalization of the mean field equation (2.5.2) which is useful in the study of periodic electroweak vortices. The material of this chapter follows closely [T4] and collects work of [BM], [BLS], [LS], [L2], [BT1], [BT2], [BCLT], [T5], and [T6].

5.2 Background material

In this section we collect some basic properties concerning solutions of the Liouville-type equation

$$-\Delta u = We^u \text{ in } \Omega, \tag{5.2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open regular domain and W a given function.

Most of the results here are stated, keeping in mind the following model weight function:

$$W(z) = |z|^{2\alpha} V(z),\ \alpha \in \mathbb{R}^+,\ V \in C(\Omega) \text{ and } 0 \in \Omega. \tag{5.2.2}$$

We start by deriving an inequality of the John–Nirenberg type and we use it to show that if $u \in L^1_{\text{loc}}(\Omega)$ satisfies (in sense of distributions)

$$-\Delta u = f \text{ in } \Omega, \tag{5.2.3}$$

and $f \in L^1_{\text{loc}}(\Omega)$, then

$$e^{|u|} \in L^p_{\text{loc}}(\Omega),\ p \geq 1. \tag{5.2.4}$$

Notice that by means of elliptic regularity theory, we get easily that $u \in W^{1,q}_{\text{loc}}(\Omega)$ for $1 \leq q < 2$. Since the power $q = 2$ is just missed, (5.2.4) cannot be deduced simply by (localizing) the Moser–Trudinger inequality.

Lemma 5.2.1 *Let $f \in L^1(\Omega)$ and $u \in W^{1,q}(\Omega)$, $1 < q < 2$ satisfy*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \leq 0 & \text{in } \partial\Omega. \end{cases} \tag{5.2.5}$$

Then for every $\delta \in (0, 4\pi)$, we have

$$\int_\Omega e^{\frac{4\pi-\delta}{\|f\|_{L^1(\Omega)}} u} \leq \frac{16\pi^2}{\delta} (\text{diam } \Omega)^2. \tag{5.2.6}$$

Proof. Let $R = \text{diam}\,\Omega$ be the diameter of Ω. So, for some ball B_R of radius R, we have $\Omega \subset B_R$. Extend $f = 0$ outside Ω and let

$$\overline{u}(z) = \frac{1}{2\pi} \int_{B_R} \log\left(\frac{2R}{|z-y|}\right) |f(y)|\, dy,$$

which satisfies

$$-\Delta\overline{u} = |f| \text{ in } \mathbb{R}^2 \text{ and } \overline{u} \geq 0 \text{ in } B_{2R}.$$

By the (weak) maximum principle, $u \leq \overline{u}$ in Ω, and so it suffices to show that (5.2.6) holds for $\overline{u}$. To this purpose, we use Jensen's inequality (2.5.10) to get

$$e^{\frac{4\pi-\delta}{\|f\|_{L^1(\Omega)}} \overline{u}} \leq \int_{B_R} \left(\frac{2R}{|z-y|}\right)^{2-\frac{\delta}{2\pi}} \frac{|f(y)|}{\|f\|_{L^1(\Omega)}}\, dy.$$

Consequently, integrating both sides of (5.2.6) over B_R we have

$$\begin{aligned}\int_{B_R} e^{\frac{4\pi-\delta}{||f||_{L^1(\Omega)}}\overline{u}}\,dx &\le (2R)^{2-\frac{\delta}{2\pi}}\int_{B_R}\int_{B_R}\frac{1}{|x-y|^{2-\frac{\delta}{2\pi}}}\frac{|f(y)|}{||f||_{L^1(\Omega)}}\,dxdy\\ &\le (2R)^{2-\frac{\delta}{2\pi}}\int_{B_R}\left(\int_{B_R}\frac{dx}{|x-y|^{2-\frac{\delta}{2\pi}}}\right)\frac{|f(y)|}{||f||_{L^1(\Omega)}}\,dy\\ &\le \frac{4\pi^2}{\delta}(2R)^2=\frac{16\pi^2}{\delta}(\mathrm{diam}\Omega)^2.\end{aligned}$$

Thus, (5.2.6) holds for $\overline{u}$, and hence for u. □

Remark 5.2.2 As a consequence of Lemma 5.2.1, we see that if $u \in W^{1,q}(\Omega)$, $1 < q < 2$ satisfies the Dirichlet problem

$$\begin{cases}-\Delta u = f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{5.2.7}$$

then (5.2.5) holds for $|u|$, and we get the formulation of (5.2.5) as given in [BM].

Lemma 5.2.3 *Let $f \in L^1(\Omega)$ and $u \in W^{1,q}(\Omega)$, $1 < q < 2$ satisfy (5.2.7), then $e^{|u|} \in L^p(\Omega)$, $\forall p \ge 1$.*

Proof. For $p \ge 1$, decompose $f = f_1 + f_2$ with $||f_1||_{L^1(\Omega)} \le \frac{4\pi-1}{p}$ and $f_2 \in L^\infty(\Omega)$. Accordingly, decompose $u = u_1 + u_2$ with u_j, $j = 1, 2$, the unique solution for the Dirichlet problem

$$-\Delta u_j = f_j \text{ in } \Omega,\ u_j = 0 \text{ in } \partial\Omega.$$

Clearly $u_2 \in L^\infty(\Omega)$, while we can apply Remark 5.2.2 to u_1 (with $\delta = 1$), to see that $e^{|u_1|} \in L^p(\Omega)$. Consequently,

$$e^{|u|} \le e^{|u_1|+|u_2|} \le ||e^{|u_2|}||_{L^\infty(\Omega)}e^{|u_1|} \in L^p(\Omega),$$

and the desired conclusion follows. □

The "local" version of Lemma 5.2.3 also holds as follows:

Corollary 5.2.4 *Let $f \in L^1_{loc}(\Omega)$ and $u \in L^1_{loc}(\Omega)$ satisfy (5.2.3) in the sense of distributions. Then (5.2.4) holds.*

Proof. By elliptic regularity thoery, $u \in W^{1,q}(\Omega)$ for $1 < q < 2$. Thus, it suffices to apply Lemma 5.2.3 to ηu, $\forall \eta \in C_0^\infty(\Omega)$. □

In other words, we see that if u and $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$, then (5.2.4) holds.

Next, we apply the results above to deduce some regularity results for solutions of (5.2.1).

Lemma 5.2.5 *Let $W \in L^p_{loc}(\Omega)$ with $p > 1$ and $u \in L^1_{loc}(\Omega)$ be such that $e^u \in L^{p'}_{loc}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. If u satisfies (5.2.1) (in the sense of distributions), then $u \in W^{2,p}_{loc}(\Omega)$. If in addition, $W \in C^{\beta}_{loc}(\Omega)$ $\beta \in (0, 1]$, then $u \in C^{2,\beta}_{loc}(\Omega)$ and it defines a classical solution for (5.2.1).*

Proof. Let $f = We^u$. By the given assumptions $f \in L^1_{\text{loc}}(\Omega)$ and therefore we can use Corollary 5.2.4 to see that $e^{|u|} \in L^p_{\text{loc}}(\Omega)$, $\forall p \geq 1$. Consequently, $f \in L^s_{\text{loc}}(\Omega)$ for some $s > 1$, and therefore we can use elliptic regularity theory to find that $u \in W^{2,s}_{\text{loc}}(\Omega)$ for some $s > 1$. In particular, $u \in L^{\infty}_{\text{loc}}(\Omega)$ and therefore $f \in L^p_{\text{loc}}(\Omega)$. In turn, $u \in W^{2,p}_{\text{loc}}(\Omega)$ as claimed. If we also have that $W \in C^{\beta}_{\text{loc}}(\Omega)$, then $f \in C^{\beta}_{\text{loc}}(\Omega)$ and the desired conclusion follows by Schauder's estimates (cf. [GT]). □

Corollary 5.2.6 *Let $u \in W^{1,2}_{loc}(\Omega)$ satisfy (5.2.1). We have:*

if $W \in L^p_{loc}(\Omega)$ for some $p > 1$, then $u \in W^{2,p}_{loc}(\Omega)$;

if $W \in C^{\beta}_{loc}(\Omega)$ for some $\beta \in (0, 1]$, then $u \in C^{2,\beta}_{loc}(\Omega)$ and u defines a classical solution for (5.2.1).

Proof. Simply observe that in this case by (localizing) the Moser–Trudinger inequality (2.4.9), we know that $e^{|u|} \in L^p_{\text{loc}}(\Omega)$, $\forall p \geq 1$. Therefore, Lemma 5.2.5 applies to u and yields to the desired conclusion. □

Remark 5.2.7 Corollary 5.2.6 can be used to justify the regularity of the various weak solutions constructed in Chapters 3, 4, and 7 for the elliptic problems arising in Chern–Simons and electroweak vortex theory. Similarly, this allows us to solve the mean field equation weakly in the Sobolev space H^1 by means of variational methods.

Next, we wish to point out a Harnack-type inequality, valid for solutions of (5.2.1) when W satisfies (5.2.2). To this purpose, we start with the following:

Proposition 5.2.8 *Let $f \in L^p(\Omega)$ for some $1 < p \leq \infty$ and u satisfy (5.2.5). For any subdomain $\Omega' \subset\subset \Omega$ there exists a constant $\beta \in (0, 1)$, depending on Ω and Ω' only, and a constant $\gamma > 0$, depending on $|\Omega|$ and p only, such that*

$$\sup_{\Omega'} u \leq \beta \inf_{\Omega'} u + (1 + \beta)\gamma \, \|f\|_{L^p(\Omega)}. \tag{5.2.8}$$

Proof. Inequality (5.2.8) is just a direct consequence of Harnack's inequality. To this purpose, let w be the unique solution for the Dirichlet problem:

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{in } \partial\Omega. \end{cases}$$

Since $f \in L^p(\Omega)$ and $p > 1$, standard elliptic estimates (see [GT]) imply that

$$\max_{\bar{\Omega}} |w| \leq \gamma \, \|f\|_{L^p(\Omega)}, \tag{5.2.9}$$

with $\gamma > 0$ a suitable constant depending on $|\Omega|$ and p only. Moreover, we see that the function $w - u$ defines an harmonic function in Ω which is nonnegative in $\partial\Omega$. So $w - u$ is nonnegative in Ω, and we can apply Harnack's inequality to obtain a constant $\beta \in (0, 1)$, depending on Ω and Ω' only, such that

$$\sup_{\Omega'}(w - u) \leq \frac{1}{\beta} \inf_{\Omega'}(w - u) \tag{5.2.10}$$

From (5.2.9) and (5.2.10), we derive

$$\sup_{\Omega'} u \leq \beta \inf_{\Omega'} u + (1 + \beta) \max_{\Omega} |w| \leq \beta \inf_{\Omega'} u + (1 + \beta)\gamma \, \|f\|_{L^p(\Omega)},$$

as claimed. □

For later use, we present the following consequence of Proposition 5.2.8.

Corollary 5.2.9 *There exists a (universal) constant $\beta \in (0, 1)$ such that, if ξ satisfies*

$$\begin{cases} -\Delta\xi = g & \text{in } B_{2R}, \\ \xi \leq C & \text{in } \partial B_{2R}, \end{cases}$$

with $g \in L^p(B_{2R})$ for some $1 < p \leq +\infty$, then

$$\sup_{B_R} \xi \leq \beta \inf_{B_R} \xi + (1 + \beta)\gamma_p R^{2(1-1/p)} \|g\|_{L^p_{B_{2R}}} + (1 - \beta)C,$$

for a suitable constant $\gamma_p > 0$ depending on p only.

Proof. Let $u(z) = \xi(Rz) - C$. It satisfies (5.2.5) in $\Omega = B_2$ with $f(z) = R^2 g(Rz) \in L^p(B_2)$. So, we can apply Proposition 5.2.8 to u with $\Omega' = B_1$ to obtain a universal constant $\beta \in (0, 1)$ and $\gamma_p > 0$ depending on p only, such that

$$\sup_{B_1} u \leq \beta \inf_{B_1} u + (1 + \beta)\gamma_p \|f\|_{L^p(B_2)}.$$

From the above inequality, we easily derive the desired conclusion. □

Proposition 5.2.10 *There exists a (universal) constant $\beta \in (0, 1)$ such that for a given $b > 0$, $\alpha \geq 0$, and $C > 0$, every solution u of (5.2.1) in $\Omega := \left\{\frac{r}{2} \leq |z| \leq 2R\right\}$, with*

$$W(z) = |z|^{2\alpha} V(z), \, \|V\|_{L^\infty} \leq b$$

and

$$\sup_{\Omega}\{u(z) + 2(\alpha + 1) \log |z|\} \leq C, \tag{5.2.11}$$

satisfies

$$\sup_{|z|=\rho} u \leq \beta \inf_{|z|=\rho} u + 2(\alpha + 1)(\beta - 1) \log \rho + c, \tag{5.2.12}$$

for every $\rho \in (r, R)$ and a suitable constant $c > 0$ depending only on α, b, and C.

Remark 5.2.11 We wish to stress once more that neither β or c depend on r and R. Furthermore, property (5.2.11) will appear as a natural condition in the sequel.

Proof of Proposition 5.2.10. For a given $\rho \in (r, R)$, let

$$v(z) = u(\rho z) + 2(\alpha + 1)\log \rho, \tag{5.2.13}$$

satisfying:

$$-\Delta v = |z|^{2\alpha} V(\rho z)e^{v} \text{ in } D := \left\{\frac{1}{2} < |z| < 2\right\}.$$

Thus, setting $f(z) = |z|^{2\alpha} V(\rho z)e^{v}$ and recalling (5.2.11) we see that

$$\sup_{\bar{D}} v \le C + 2(\alpha + 1)\log 2 := C_1$$

and

$$\|f\|_{L^\infty(D)} \le 4be^{C}.$$

Therefore, we can apply Proposition 5.2.8 to $v - C_1$ in $\Omega' = \{z : |z| = 1\} \subset D$ to obtain (universal) constants $\beta \in (0, 1)$ and $\gamma > 0$, such that

$$\sup_{|z|=1} v \le \beta \inf_{|z|=1} v + (1+\beta)\gamma \|f\|_{L^\infty(D)} + (1-\beta)C_1.$$

From the inequality above and by means of (5.2.13) we immediately derive (5.2.12). □

We conclude this section with a useful Pohozaev-type identity valid for (smooth) solutions of (5.2.1):

Pohozaev's identity: Let $W \in W^{1,\infty}(\Omega)$ and $u \in C^2(\Omega)$ satisfy (5.2.1). The following identity holds for every regular subdomain $D \subseteq \Omega$:

$$\begin{aligned}\int_{\partial D}\left(z\cdot\nu\frac{|\nabla u|^2}{2} - (\nu\cdot\nabla u)(z\cdot\nabla u)\right)d\sigma &= \int_{\partial D} z\cdot\nu W e^{u} d\sigma \\ &\quad - \int_D (2W + z\cdot\nabla W)e^{u}\end{aligned} \tag{5.2.14}$$

where ν is the outward normal vector to ∂D.

Proof. As usual in deriving Pohozaev-type identities, we multiply equation (5.2.1) by $z \cdot \nabla u$ an integrate over D to obtain

$$-\int_D (z\cdot\nabla u)\Delta u = \int_D W e^{u} z\cdot\nabla u. \tag{5.2.15}$$

We shall expand each side of (5.2.15). In fact, by direct inspection, it is not difficult to verify the identity

$$\Delta u(z\cdot\nabla u) = \operatorname{div}\,(\nabla u(z\cdot\nabla u)) - \operatorname{div}\left(z\frac{|\nabla u|^2}{2}\right),$$

and (via the Green–Gauss theorem) obtain the left-hand side of (5.2.14).

Concerning the right-hand side of (5.2.14), we find

$$\int_D We^u z\cdot\nabla u = \int_D Wz\cdot\nabla e^u = \int_D \operatorname{div}(zWe^u) - 2\int_D We^u - \int_D (z\cdot\nabla W)e^u$$
$$= \int_{\partial D}(z\cdot\nu)We^u d\sigma - 2\int_D We^u - \int_D (z\cdot\nabla W)e^u,$$

and (5.2.14) is established. □

In the special case where W is given by (5.2.2), we can further expand (5.2.14) and conclude:

Corollary 5.2.12 *Let $u \in C^2(B_1)$ satisfy (5.2.1) in B_1 where (5.2.2) holds with $V \in W^{1,+\infty}(B_1)$. Then, for every $r \in (0,1)$, we have*

$$\begin{aligned} r\int_{\{|z|=r\}}\left(\frac{1}{2}|\nabla u|^2 - (\nu\cdot\nabla u)^2\right)d\sigma - r^{2\alpha+1}\int_{\{|z|=r\}} Ve^u d\sigma \\ = -2(\alpha+1)\int_{\{|z|\le r\}}|z|^{2\alpha}Ve^u - \int_{\{|z|\le r\}}|z|^{2\alpha}(z\cdot\nabla V)e^u. \end{aligned} \tag{5.2.16}$$

5.3 Basic analytical facts

Here, we aim to derive some preliminary facts concerning a sequence u_k satisfying

$$-\Delta u_k = W_k e^{u_k} \text{ in } \Omega, \tag{5.3.1}$$

where W_k is a family of weight functions and $\Omega \subset \mathbb{R}^2$ is a bounded open regular domain.

An important starting point for our discussion is given by the following:

Proposition 5.3.13 *Let u_k satisfy (5.3.1) and assume that*

(i) $\|W_k\|_{L^\infty(\Omega)} + \|u_k^+\|_{L^1(\Omega)} \le C$, *for suitable* $C > 0$,

(ii) $\limsup_{k\to+\infty}\int_\Omega |W_k|e^{u_k} < 4\pi$.

Then u_k^+ is uniformly bounded in $L^\infty_{loc}(\Omega)$.

This result has been obtained by Brezis–Merle in [BM], and holds within a more general L^p-framework, where assumptions (i) and (ii) are replaced by

$$\|W_k\|_{L^p(\Omega)} + \|u_k^+\|_{L^1(\Omega)} \le C \text{ and } \limsup_{k\to+\infty}\int_\Omega |W_k|e^{u_k} < \frac{4\pi}{p'},$$

with $1 < p \le +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. In any regular subdomain $D \subset\subset \Omega$, decompose

$$u_k = u_{1,k} + u_{2,k},$$

where $u_{1,k}$ and $u_{2,k}$ are uniquely defined as satisfying the following Dirichlet problems:

$$\begin{cases} -\Delta u_{1,k} = 0 & \text{in } D, \\ u_{1,k} = u_k & \text{on } \partial D, \end{cases}$$

and

$$\begin{cases} -\Delta u_{2,k} = W_k e^{u_k} & \text{in } D, \\ u_{2,k} = 0 & \text{on } \partial D. \end{cases}$$

For some $\varepsilon_0 > 0$ and $k_0 \in \mathbb{N}$, we have $\int_D W_k e^{u_k} \leq 4\pi - \varepsilon_0, \forall k \geq k_0$. Therefore, we can apply Lemma 5.2.1 and Remark 5.2.2 to $u_{2,k}$ to conclude

$$\|e^{|u_{2,k}|}\|_{L^p(D)} \leq C,$$

for suitable $p > 1$ and $C > 0$.

In particular, from the estimate above, it follows that $u_{2,k}$ is uniformly bounded in $L^1(D)$. Since $u_{1,k}^+ \leq u_k^+ + |u_{2,k}|$, we also get a uniform bound for $u_{1,k}^+$ in $L^1(D)$. The mean value theorem then implies that $u_{1,k}^+$ is actually uniformly bounded in $L^\infty_{\text{loc}}(D)$.

Moreover, we conclude that $W_k e^{u_k}$ is uniformly bounded in $L^p_{\text{loc}}(D)$, for suitable $p > 1$. Consequently, $u_{2,k}$ is uniformly bounded in $L^\infty_{\text{loc}}(D)$ and the desired conclusion follows. □

Following [BM] we give the following:

Definition 5.3.14 A point $z_0 \in \Omega$ is called a *blow-up point* for the sequence u_k in Ω, if there exists a sequence $\{z_k\} \subset \Omega$ such that

$$z_k \to z_0, \text{ and } \lim_{k \to +\infty} u_k(z_k) = +\infty.$$

In the sequel, we shall denote by S the set of blow-up points, and refer to it as the *blow-up set*.

As a consequence of Proposition 5.3.13, we find:

Corollary 5.3.15 *Suppose that u_k satisfies (5.3.1) with W_k such that*

$$\|W_k\|_{L^\infty(\Omega)} + \int_\Omega \frac{1}{|W_k|^q} \leq C, \text{ for some } q > 0. \tag{5.3.2}$$

(i) If

$$\limsup_{k \to +\infty} \int_\Omega |W_k| e^{u_k} < 4\pi, \text{ then } u_k^+ \text{ is uniformly bounded in } L^\infty_{loc}(\Omega). \tag{5.3.3}$$

(ii) If $z_0 \in \Omega$ is a blow-up point for u_k, then

$$\liminf_{k\to+\infty} \int_{B_\delta(z_0)} |W_k|e^{u_k} \geq 4\pi, \tag{5.3.4}$$

for every $\delta > 0$ sufficiently small.

Furthermore, if, for a suitable constant $C > 0$, we have

$$\int_\Omega |W_k|e^{u_k} < C, \tag{5.3.5}$$

then u_k can admit only a finite number of blow-up points in Ω.

Proof. In proving (i) and (ii) we can always assume that (5.3.5) holds, as otherwise the conclusion in (ii) is obvious. Let $t = \frac{q}{q+1} \in (0, 1)$, and observe that

$$t\int_\Omega u_k^+ \leq \int_\Omega e^{tu_k} \leq \left(\int_\Omega |W_k(x)|e^{u_k}\right)^t \left(\int_\Omega \frac{1}{|W_k(x)|^q}\right)^{1-t} \leq C. \tag{5.3.6}$$

Hence, we can use Proposition 5.3.13 to yield (5.3.3). To establish (ii), we use the estimate (5.3.6), with the integrals taken over $B_\delta(z_0)$, $\delta > 0$. In this way, we can check the validity of the assumption (i) of Proposition 5.3.13 in $B_\delta(z_0)$, and conclude that if $z_0 \in \Omega$ is a blow-up point for u_k, (hence it is a blow-up point for any of its subsequences), then necessarily

$$\liminf_{k\to+\infty} \int_{B_\delta(z_0)} |W_k(x)|e^{u_k} \geq 4\pi.$$

Therefore, when (5.3.5) holds, only a finite number of such blow-up points are allowed. □

Remark 5.3.16 To justify the nature of assumptions (5.3.2), note that when

$$0 < b_1 \leq |W_k| \leq b_2 \text{ in } \Omega, \tag{5.3.7}$$

then (5.3.2) certainly holds and condition (5.3.5) can be replaced by the equivalent condition

$$\int_\Omega e^{u_k} \leq C. \tag{5.3.8}$$

In fact, the results stated here are available in [BM] under the assumptions (5.3.7) and (5.3.8). On the other hand, since we are motivated by problem (5.1.1), (5.1.2), we actually wish to cover the case where W_k vanishes at a finite number of points with finite order. In this situation, (5.3.2) and (5.3.5) furnish the natural replacement of (5.3.7) and (5.3.8).

As a consequence of Proposition 5.4.19 we have:

Proposition 5.3.17 *Let u_k satisfy (5.3.1) in Ω and assume (5.3.2) and (5.3.5). There exists a subsequence $\{u_{n_k}\}$ of $\{u_k\}$ such that $u_{n_k}^+$ is uniformly bounded in $L^\infty_{loc}(\Omega \setminus S)$, where $S \subset \Omega$ is the blow-up set relative to u_{n_k}.*

Proof. Along a subsequence, (denoted the same way) we can assume that $|W_k|e^{u_k} \rightharpoonup \nu$, weakly in the sense of measure in Ω, where ν is a finite measure in Ω.

Set

$$\Sigma = \{z_0 \in \Omega : \nu(\{z_0\}) \geq 4\pi\}, \tag{5.3.9}$$

and observe that necessarily Σ is a finite set. By (i) in Corollary 5.3.15, we know that u_k^+ is uniformly bounded in $L^\infty_{\text{loc}}(\Omega \setminus \Sigma)$. So, the blow-up set S_1 of u_k in Ω is contained in Σ. We claim that Σ coincides with the blow-up set of a possible subsequence u_{n_k}, for which the desired conclusion holds. Indeed, if there exists $z_0 \in \Sigma \setminus S_1$, we let $\delta > 0$ sufficiently small, so that $\bar{B}_\delta(z_0) \cap S_1$ is empty and z_0 is the only point of Σ in $B_\delta(z_0)$.

Note that

$$\max_{B_\delta(z_0)} u_k^+ \to +\infty, \text{ as } k \to +\infty.$$

In fact, if on the contrary, (along a subsequence) we suppose $\|u_k^+\|_{L^\infty(B_\delta(z_0))} \leq C$, then for every $\varepsilon \in (0, \delta)$, we have

$$\int_{B_\varepsilon(z_0)} |W_k|e^{u_k} = O(\varepsilon^2) \to 0, \text{ as } \varepsilon \to 0,$$

in contradiction to the fact that $z_0 \in \Sigma$.

Thus, taking a subsequence, we can let $z_k \in \overline{B_\delta(z_0)} : u_k(z_k) = \max_{B_\delta(z_0)} u_k \to +\infty$, and $z_k \to z_1 \in \overline{B_\delta(z_0)}$.

Since, u_k^+ is uniformly bounded in $L^\infty_{\text{loc}}(\Omega \setminus \Sigma)$, then $z_1 = z_0$ necessarily, and so z_0 is a blow-up point for such a *new* subsequence, whose blow-up set S_2 contains the set $S_1 \cup \{z_0\}$. We are finished when $\Sigma = S_2$. Otherwise, we iterate the argument above to obtain a new subsequence whose blow-up set contains an additional point of Σ. Since the number of elements of Σ is finite, this procedure must stop after a finite number of steps, for which we arrive at the desired subsequence $\{u_{n_k}\}$, having Σ as blow-up set. □

Our next task is to investigate the nature of the limiting measure ν of the sequence $W_{n_k}e^{u_{n_k}}$ in relation to the blow-up set S. In this direction, a rather complete analysis is available for the non-vanishing case (namely, when (5.3.7) and (5.3.8) hold) through the work of Brezis–Merle [BM], Li–Shafrir [LS], Brezis–Li–Shafrir [BLS], Li [L2], and Chen–Lin [ChL1] and [ChL2]. But for our applications to the vortex problem, we need to consider the situation where W_k actually vanishes at a blow-up point, in other words, when a blow-up point and a vortex point coincide, with say the origin. With this in mind, for given $\alpha_k > 0$, we take the function W_k of the form, $W_k(z) = |z|^{2\alpha_k} V_k(z)$, and devote the next section to the corresponding "local" analysis around the origin.

5.4 A concentration-compactness principle

In this section we wish to investigate the behavior of a sequence u_k satisfying

$$-\Delta u_k = |z|^{2\alpha_k} V_k e^{u_k} \text{ in } B_1(0), \tag{5.4.1}$$

for the case when it admits a blow-up point at the origin. We assume that

$$0 < b_1 \leq V_k \leq b_2 \text{ in } B_1(0) \tag{5.4.2}$$

$$\alpha_k \geq 0 \text{ and } \alpha_k \to \alpha \text{ as } k \to +\infty. \tag{5.4.3}$$

Since under the assumptions of (5.4.2) and (5.4.3), the function $W_k = |z|^{2\alpha} V_k$ satisfies (5.3.2) (with sufficiently small $q > 0$), we know already by the results of the previous section that the following holds:

Corollary 5.4.18 *Let u_k satisfy (5.4.1) and suppose that (5.4.2) and (5.4.3) hold.*
(i) If $\limsup_{k\to+\infty} \int_{B_1} |z|^{2\alpha_k} V_k(z) e^{u_k} < 4\pi$, then u_k^+ is uniformly bounded in $L^\infty_{loc}(B_1(0))$.
(ii) If zero is a blow-up point for u_k, then

$$\liminf_{k\to+\infty} \int_{B_\delta} |z|^{2\alpha_k} V_k(z) e^{u_k} \geq 4\pi, \tag{5.4.4}$$

for any $\delta \in (0, 1]$.
In addition, if

$$\int_{B_{\delta_0}(0)} |z|^{2\alpha_k} V_k e^{u_k} \leq C, \text{ for some } \delta_0 \in (0, 1], \tag{5.4.5}$$

with a suitable constant $C > 0$, then zero is the only blow-up point for u_k in $B_\delta(0)$, for suitable, sufficiently small $\delta \in (0, \delta_0)$.

Observe that problem (5.4.1) enjoys a nice scale-invariance property as follows:

$$\begin{aligned} &\text{if } u_k \text{ satisfies (5.4.1), then for any } \lambda > 0, \text{ the function} \\ &u_{k,\lambda}(z) = u_k\left(\frac{z}{\lambda}\right) + 2(\alpha+1)\log\frac{1}{\lambda} \\ &\text{satisfies problem (5.4.1) in } B_\lambda = \{z : |z| < \lambda\}, \text{ with } V_k \text{ replaced by} \\ &V_{k,\lambda}(z) := V_k\left(\frac{z}{\lambda}\right) \text{ and } \int_{B_1} |z|^{2\alpha_k} V_k e^{u_k} = \int_{B_\lambda} |z|^{2\alpha_k} V_{k,\lambda} e^{u_{k,\lambda}}. \end{aligned} \tag{5.4.6}$$

By means of such invariance, we will be able to use a blow-up technique to improve and complete the result above in various directions.

5.4.1 The blow-up technique

We start by showing how to improve (5.4.4) under the additional condition:

$$V_k \to V \text{ uniformly in } C^0_{\text{loc}}. \tag{5.4.7}$$

Proposition 5.4.19 *Let u_k satisfy (5.4.1) and assume that (5.4.2), (5.4.3), and (5.4.7) hold. If zero is a blow-up point for u_k, then*

$$\liminf_{k\to+\infty}\int_{B_\delta(0)}|z|^{2\alpha_k}V_k(z)e^{u_k}\geq 8\pi,\ \forall\delta\in(0,1]. \tag{5.4.8}$$

Proposition 5.4.19 should be compared with an equivalent result established by Li–Shafrir in [LS] for solution-sequences $\{u_k\}$ of (5.3.1) and satisfying:

$$W_k\to W \text{ uniformly in } C_{\text{loc}}(\Omega) \text{ and } \int_\Omega e^{u_k}\leq C. \tag{5.4.9}$$

Proposition 5.4.20 *([LS]). Let u_k satisfy (5.3.1) and assume (5.4.9). If $z_0\in\Omega$ is a blow-up point for u_k, then*

$$\liminf_{k\to+\infty}\int_{B_\delta(z_0)}W_ke^{u_k}\geq 8\pi,\ \forall\delta>0 \textit{ small.} \tag{5.4.10}$$

Note that the second condition in (5.4.9) permits us to verify assumption (i) of Proposition 5.3.13; and this implies that (5.3.4) holds for u_k. Hence, Propositions 5.4.19 and 5.4.20 aim to improve the lower bound in (5.4.4) and (5.3.4), respectively, when the sequence of weight functions admits a uniform limit. Clearly, Proposition 5.4.20 covers as a particular case Proposition 5.4.19, when the limiting function W in (5.4.9) satisfies $0<b_1\leq|W|\leq b_2$ in Ω. The lower bound 8π is sharp for *both* (5.4.8) and (5.4.10), as one sees from the examples given in Section 5.5.5 below.

To derive Proposition 5.4.19, we shall need the following preliminary result:

Lemma 5.4.21 *Let $R_k\to+\infty$, $y_k\to z_0$ and $C>0$. Assume that ξ_k satisfies*

$$\begin{cases}-\Delta\xi_k=U_k(z)e^{\xi_k} \text{ in } D_k=\{|z|<R_k\},\\ \xi_k(y_k)=0,\\ \sup\limits_{D_k}\xi_k+\displaystyle\int_{D_k}|U_k|e^{\xi_k}\leq C,\end{cases}$$

where

$$U_k\to\mu|z|^{2a} \text{ uniformly in } C^0_{loc}(\mathbb{R}^2), \text{ with } \mu>0 \text{ and } a\geq 0.$$

Then the following holds:
(a) ξ_k is uniformly bounded in $L^\infty_{loc}(\Omega)$;
(b) along a subsequence,

$$\xi_k\to\xi \text{ uniformly in } C^{2,\beta}_{loc}(\mathbb{R}^2),\ \beta\in(0,1) \tag{5.4.11}$$

and ξ satisfies

$$\begin{cases}-\Delta\xi=\mu|z|^{2a}e^{\xi} \text{ in } \mathbb{R}^2,\\ \displaystyle\int_{\mathbb{R}^2}|z|^{2a}e^{\xi}<+\infty,\end{cases} \tag{5.4.12}$$

with $\xi(z_0)=0$.

Remark 5.4.22 1) Obviously the condition $\xi_k(y_k) = 0$ is a "normalization" condition, and in fact, the conclusion above follows under the more general requirement: $\limsup_{k\to+\infty} |\xi_k(y_k)| < +\infty$.
2) We can apply the classification result in Corollary 2.2.2, i.e., (2.2.17) with $\alpha = a$, to see that ξ in (5.4.12) takes the form:

$$\xi(z) = \log\left(\frac{\lambda_0}{(1+\lambda_0\mu\gamma_a|z^{a+1}-y_0|^2)^2}\right), \ \gamma_a = \frac{1}{8(1+a)^2} \tag{5.4.13}$$

with

$$\mu\int_{\mathbb{R}^2} |z|^{2a}e^{\xi} = 8\pi(1+a); \tag{5.4.14}$$

in addition,

$$\text{if } a \in (0,+\infty)\setminus\mathbb{N}, \text{ then } y_0 = 0 \text{ necessarily.} \tag{5.4.15}$$

Moreover, in (5.4.13) the parameters $y_0 \in \mathbb{C}$ and $\lambda_0 \in \mathbb{R}^+$ are constrained by the condition $\xi(z_0) = 0$. This implies:

$$s_0 := \gamma_a\mu|z_0^{a+1} - y_0| \le \frac{1}{4} \text{ and } \lambda_0 = \frac{1}{2s_0^2}\left(1 - 2s_0 \pm \sqrt{1-4s_0}\right) \ge 1. \tag{5.4.16}$$

At times, it is known that $\xi(z_0) = \max_{\mathbb{R}^2}\xi = 0$. For such cases, (5.4.13) holds with $y_0 = z_0^{a+1}$ and $\lambda = 1$.

Proof of Lemma 5.4.21. Let $f_k = U_k e^{\xi_k}$, so that according to our assumptions, f_k is uniformly bounded in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. Furthermore, for every $R > 0$, we can use Corollary 5.2.4, to obtain $\beta \in (0,1)$, independent of R, such that

$$\sup_{B_R}\xi_k \le \beta\inf_{B_R}\xi_k + C_R, \tag{5.4.17}$$

for suitable $C_R > 0$.

Since for R sufficiently large, $\sup_{B_R}\xi_k \ge \xi_k(y_k) = 0$, we can use (5.4.17) to ensure that ξ_k is also bounded from below in B_R, and uniformly so in k. In other words, ξ_k is uniformly bounded in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. We can then use standard elliptic regularity theory to extend such uniform bounds to hold in $C^{2,\gamma}_{\text{loc}}(\mathbb{R}^2)$, for some $\gamma \in (0,1)$.

Hence, by a diagonal process, we obtain a subsequence (denoted in the same way) such that

$$\xi_k \to \xi, \text{ uniformly in } C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2), \tag{5.4.18}$$

with ξ satisfying (5.4.12) and $\xi(z_0) = 0$ as claimed. □

To simplify notation and without loss of generality, from now on we suppose that (5.4.7) is satisfied with

$$V(0) = 1. \tag{5.4.19}$$

Proof of Proposition 5.4.19. To establish (5.4.8), we only need to consider the case where (5.4.5) holds. Thus, for $\delta > 0$ sufficiently small, we can assume that zero is the only blow-up point for u_k in $\bar{B}_\delta$.

Letting $z_k \in \bar{B}_\delta$,

$$u_k(z_k) = \max_{\bar{B}_\delta} u_k, \tag{5.4.20}$$

we if necessary, take a subsequence so that,

$$u_k(z_k) \to +\infty \text{ and } z_k \to 0. \tag{5.4.21}$$

Set

$$\varepsilon_k = e^{-\frac{u_k(z_k)}{2(\alpha_k+1)}} \to 0.$$

We now distinguish two cases:
Case 1:

$$\left|\frac{z_k}{\varepsilon_k}\right| = O(1), \text{ as } k \to +\infty. \tag{5.4.22}$$

In this case, let us assume that

$$y_k := \frac{z_k}{\varepsilon_k} \to z_0, \tag{5.4.23}$$

holds along a subsequence. Set

$$\xi_k(z) = u_k(\varepsilon_k z) + 2(\alpha_k + 1)\log \varepsilon_k. \tag{5.4.24}$$

Note that, $\xi_k(y_k) = \max\limits_{\{|z| \le \frac{\delta}{\varepsilon_k}\}} \xi_k = 0$. Thus, we easily check that ξ_k satisfies all assumptions of Lemma 5.4.21, with y_k in (5.4.23), $R_k = \frac{\delta}{\varepsilon_k} \to +\infty$, and $U_k(z) = |z|^{2\alpha_k} V_k(\varepsilon_k z) \to |z|^{2\alpha}$ uniformly in $C^0_{\text{loc}}(\mathbb{R}^2)$, as $k \to \infty$.

Consequently,

$$\xi_k \text{ is uniformly bounded in } L^\infty_{\text{loc}}(\mathbb{R}^2) \tag{5.4.25}$$

and along a subsequence,

$$\xi_k \to \xi \text{ uniformly in } C^2_{\text{loc}}(\mathbb{R}^2) \tag{5.4.26}$$

with ξ satisfying (1.3.41) for $\mu = 1$ and $a = \alpha$. In particular, ξ satisfies: $\int_{\mathbb{R}^2} |z|^{2\alpha} e^{\xi} = 8\pi(1+\alpha)$ (see (5.4.14)).

Therefore, by Fatou's lemma, we find:

$$\lim_{k\to+\infty} \int\limits_{\{|z|\le\delta\}} |z|^{2\alpha_k} V_k(z) e^{u_k} = \lim_{k\to+\infty} \int\limits_{\{|z|\le\delta/\varepsilon_k\}} U_k e^{\xi_k} \ge \int\limits_{\mathbb{R}^2} |z|^{2\alpha} e^{\xi} = 8\pi(1+\alpha),$$

and the desired conclusion follows in this case.

Remark 5.4.23 Notice that in view of (5.4.25), when (5.4.22) holds, then $\max_{\bar{B}_\delta} u_k - u_k(0) = -\xi_k(0) = O(1)$. That is, $u_k(0) = \max_{\bar{B}_\delta} u_k + O(1)$, in this case.

Case 2: $\frac{|z_k|}{\varepsilon_k} \to +\infty,$ as $k \to +\infty$.

In this situation, set $\tau_k = \frac{e^{-\frac{u_k(z_k)}{2}}}{|z_k|^{\alpha_k}} = \varepsilon_k \left(\frac{\varepsilon_k}{|z_k|}\right)^{\alpha_k} \to 0,$ as $k \to +\infty$. Let

$$\xi_k(z) = u_k(z_k + \tau_k z) - u_k(z_k)$$

and

$$U_k(z) = \left|\frac{z_k}{|z_k|} + \frac{\tau_k}{|z_k|} z\right|^{2\alpha_k} V_k(z_k + \tau_k z)$$

in $D_k = \{|z| \le \frac{\delta}{2\tau_k}\}$.

Then,

$$\begin{cases} -\Delta\xi_k = U_k(z)e^{\xi_k} \quad \text{in } D_k, \\ \xi_k(0) = 0 = \max_{D_k} \xi_k, \\ \int_{D_k} U_k(z)e^{\xi_k} \le C, \end{cases}$$

for large k. Since $\frac{\tau_k}{|z_k|} \to 0$, we see that $U_k(z) \to 1$ uniformly in $C^0_{\text{loc}}(\mathbb{R}^2)$. Therefore, we can apply Lemma 5.4.21 to ξ_k with $a = 0$, and by Remark 5.4.22, conclude that along a subsequence,

$$\xi_k \to \xi(z) = \log \frac{1}{\left(1 + \frac{1}{8}|z|^2\right)^2} \text{ uniformly in } C^2_{\text{loc}}(\mathbb{R}^2), \text{ and } \int_{\mathbb{R}^2} e^{\xi} = 8\pi.$$

Consequently,

$$\lim_{k\to+\infty} \int_{B_\delta(0)} |z|^{2\alpha_k} V_k e^{u_k} \ge \lim_{k\to+\infty} \int_{\{|z-z_k|\le\delta/2\}} |z|^{2\alpha_k} V_k e^{u_k} \ge \lim_{k\to+\infty} \int_{D_k} U_k e^{\xi_k}$$
$$\ge \int e^{\xi} = 8\pi.$$

Therefore (5.4.8) holds in this case as well.

For completeness, we also include:

Proof of Proposition 5.4.20. We proceed similarly, and let $\delta > 0$ sufficiently small such that z_0 is the only blow-up point for u_k in $\overline{B_{2\delta}(z_0)}$. By a translation, we can always assume that $z_0 = 0$. Let $z_k \in \overline{B_{2\delta}(0)}$ such that $u_k(z_k) = \max_{\{|z|<2\delta\}} u_k$. We have $z_k \to 0$ and $u_k(z_k) \to +\infty$. Setting

$$\varepsilon_k = e^{-\frac{u_k(z_k)}{2}} \to 0, \text{ as } k \to +\infty$$

and

$$\xi_k(z) = u_k(z_k + \varepsilon_k z) + 2\log\varepsilon_k, \ z \in D_k = \left\{|z| < \frac{\delta}{\varepsilon_k}\right\}$$

we see that

$$\begin{cases} -\Delta\xi_k = U_k e^{\xi_k} \text{ in } D_k, \\ \xi_k(0) = \max_{D_k} \xi_k = 0 \text{ and } \int_{D_k} e^{\xi_k} \leq C, \end{cases}$$

with $U_k(z) = W_k(z_k + \varepsilon_k z) \to W(0)$ uniformly in C^0_{loc}. Exactly as above, such conditions suffice to find a subsequence (denoted in the same way), such that $\xi_k \to \xi$ uniformly in $C^{2,\alpha}_{\text{loc}}$ and ξ satisfies:

$$\begin{cases} -\Delta\xi = W(0)e^{\xi}, \\ \xi(0) = \max_{\mathbb{R}^2} \xi = 0, \\ \int_{\mathbb{R}^2} e^{\xi} < +\infty. \end{cases}$$

Since $\Delta\xi(0) \leq 0$, we see that necessarily $W(0) \geq 0$. On the other hand, $W(0) = 0$ would imply $\xi =$ constant, in contradiction with the integratability of e^{ξ} in $\mathbb{R}^2$. Therefore,

$$\mu = W(0) > 0 \text{ and } \xi(z) = \log\left(\frac{1}{\left(1 + \frac{\mu}{8}|z|^2\right)^2}\right).$$

In particular,

$$\liminf_{k\to+\infty} \int_{B_\delta(0)} W_k e^{u_k} = \liminf_{k\to+\infty} \int_{D_k} U_k e^{\xi_k} \geq W(0) \int_{\mathbb{R}^2} e^{\xi} = 8\pi,$$

and (5.4.9) is established.

The proof of Proposition 5.4.19 leaves us to wonder when Case 1 actually occurs. More generally, when can we replace (5.4.8) with the improved lower bound

$$\liminf_{k\to+\infty} \int_{B_\delta(0)} |z|^{2\alpha_k} V_k e^{u_k} \geq 8(1+\alpha), \tag{5.4.27}$$

for every $\delta \in (0, 1]$?

In this respect, a first simple observation points towards a condition that plays a relevant role in the sequel. □

Corollary 5.4.24 *In addition to the assumptions of Proposition 5.4.19, suppose that*

$$\sup_{|z|\leq\delta_0} \{u_k(z) + 2(\alpha_k + 1)\log|z|\} \leq C, \tag{5.4.28}$$

for suitable $\delta_0 \in (0, 1)$ and $C > 0$. Then (5.4.27) holds.

Furthermore, if (5.4.5) is also satisfied, then

$$u_k(0) = \max_{|z|\leq\delta_0} u_k + O(1) \tag{5.4.29}$$

Proof. Let $z_k \in \bar{B}_{\delta_0} : u_k(z_k) = \max_{|z| \le \delta_0} u_k \to +\infty$ and $\varepsilon_k = e^{-\frac{u_k(z_k)}{2(\alpha_k+1)}}$.

It suffices to observe that in view of (5.4.28), $z_k \to 0$ necessarily and (5.4.22) holds. This immediately yields to (5.4.27). Moreover, if we recall Remark 5.4.23, then we also conclude (5.4.29). □

When (5.4.28) fails to hold for every $\delta_0 \in (0, 1)$, we can refine Corollary 5.4.24 and Proposition 5.4.19 as follows:

Lemma 5.4.25 *Let u_k satisfy (5.4.1) and assume that (5.4.2) and (5.4.3) hold. Suppose there exists a sequence $\{z_k\} \subset B_1 \setminus \{0\}$ such that*

$$z_k \to 0, \quad u_k(z_k) + 2(\alpha_k + 1)\log|z_k| \to +\infty,$$

then

$$\liminf_{k\to+\infty} \int_{B_{\delta|z_k|}(z_k)} |z|^{2\alpha_k} V_k e^{u_k} \ge 4\pi, \tag{5.4.30}$$

$\forall\, \delta > 0$. *If in addition we assume (5.4.7), then*

$$\liminf_{k\to+\infty} \int_{B_{\delta|z_k|}(z_k)} |z|^{2\alpha_k} V_k e^{u_k} \ge 8\pi, \tag{5.4.31}$$

$\forall\, \delta > 0$.

Proof. It suffices to prove (5.4.30) and (5.4.31) for $\delta \in (0, 1)$. To this purpose, let k be sufficiently large, such that

$$u_{1,k}(z) = u_k(z_k + |z_k|z) + 2(\alpha_k + 1)\log(|z_k|)$$

is well-defined in B_1 and satisfies

$$-\Delta u_{1,k}(z) = |z|^{2\alpha_k} V_{1,k}(z) e^{u_{1,k}} \text{ in } B_1,$$

with $V_{1,k}(z) = \left|\frac{z_k}{|z_k|} + z\right|^{2\alpha_k} V_k(z_k + |z_k|z)$. Notice that zero defines a blow-up point for $u_{1,k}$ since we have

$$u_{1,k}(0) = u_k(z_k) + 2(\alpha_k + 1)\log|z_k| \to +\infty, \text{ as } k \to +\infty.$$

Since $V_{1,k}$ is uniformly bounded from above and from below away from zero in $B_\delta(0)$, $\forall\ \delta \in (0, 1)$, we can apply Corollary 5.4.18 (ii) to $u_{1,k}$ and conclude that $\liminf_{k\to+\infty} \int_{B_\delta(0)} V_{1,k} e^{u_{1,k}} \ge 4\pi, \forall \delta \in (0, 1)$.

A simple change of variables yields to (5.4.30). On the other hand, when (5.4.7) holds, we can apply Proposition 5.4.19 to any subsequence of $u_{1,k}$ and similarly derive (5.4.31). □

At this point, it appears clear that in order to analyze the behavior of u_k around the blow-up point zero, we must analyze separately the case where (5.4.28) holds, and the case where it fails.

For this purpose, it is useful to have available the following alternative:

Proposition 5.4.26 *Let u_k satisfy (5.4.1), and assume that (5.4.2), (5.4.3), and (5.4.5) hold. There exist constants $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$ and $C > 0$, such that, along a subsequence the following alternative holds:*
(i) either

$$\sup_{0<|z|\le 2\varepsilon_0} \{u_k(z) + 2(\alpha_k + 1)\log|z|\} \le C; \tag{5.4.32}$$

(ii) or there exists a sequence $\{z_k\} \subset B_1 \setminus \{0\}$ such that

$$z_k \to 0, \quad u_k(z_k) + 2(\alpha_k + 1)\log|z_k| \to +\infty, \tag{5.4.33}$$

$$\sup_{0<|z|\le 2\varepsilon_0|z_k|} \{u_k + 2(\alpha_k + 1)\log|z|\} \le C. \tag{5.4.34}$$

Proof. Taking into account (5.4.5), set

$$\beta = \limsup_{k\to+\infty} \int_{B_{\delta_0}} |z|^{2\alpha_k} V_k e^{u_k}. \tag{5.4.35}$$

If (5.4.32) fails to hold for every $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$, then we find a sequence $z_{1,k} \to 0$ and a subsequence of u_k, which we denote in the same way, such that

$$u_k(z_{1,k}) + 2(\alpha_k + 1)\log|z_{1,k}| \to +\infty.$$

Thus, by Lemma 5.4.25 we know that:

$$\liminf_{k\to+\infty} \int_{B_{\delta|z_{1,k}|}(z_{1,k})} |z|^{2\alpha_k} V_k e^{u_k} \ge 4\pi,$$

$\forall\, \delta > 0$. Repeat the alternative above for the sequence:

$$u_{1,k}(z) = u_k(|z_{1,k}|z) + 2(\alpha_k + 1)\log|z_{1,k}|.$$

Therefore, either (i) holds for suitable $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$, and leads to (5.4.33), (5.4.34) with $z_k = z_{1,k}$; or there exists a second sequence $z_{2,k} \in B_1 \setminus \{0\}$, such that, along a subsequence, there holds

$$\left|\frac{z_{2,k}}{z_{1,k}}\right| \to 0, \quad u_k(z_{2,k}) + 2(\alpha_k + 1)\log|z_{2,k}| \to +\infty,$$

$$\liminf_{k\to+\infty} \int_{B_{\delta|z_{2,k}|}(z_{2,k})} |z|^{2\alpha_k} V_k e^{u_k} \ge 4\pi, \quad \forall\, \delta > 0.$$

Note that the sets $B_{\delta|z_{1,k}|}(z_{1,k})$ and $B_{\delta|z_{2,k}|}(z_{2,k})$ do not intersect each other for $\delta \in (0,1)$ and k large.

Therefore, in this case we see that β in (5.4.35) must satisfy: $\beta \geq 8\pi$. We may also repeat the alternative above for the second iterated sequence $u_{2,k}(z) = u_k(|z_{2,k}|z) + 2(\alpha_k + 1)\log|z_{2,k}|$, and so on. Observe that each time such an iterated sequence fails to verify (5.4.32), $\forall \varepsilon_0 \in \left(0, \frac{1}{2}\right)$, we contribute with an amount of (at least) 4π to the value β in (5.4.35). Thus, after a finite number of steps, we must end up with an iterated sequence that satisfies (5.4.32) for some $\varepsilon_0 \in (0, 1/2)$. This fact, when expressed for the original sequence u_k, yields to the desired properties of (5.4.33) and (5.4.34). □

5.4.2 A Concentration-Compactness result around a "singular" point

A more elaborate answer to the question concerning the validity of (5.4.27) requires the introduction of the following suitable boundary conditions on u_k:

$$\sup_{\partial B_{\delta_0}} u_k - \inf_{\partial B_{\delta_0}} u_k \leq C, \quad \delta_0 \in (0,1], \tag{5.4.36}$$

for suitable $C > 0$. As we shall see, the behavior of u_k around the origin (a blow-up point) is seriously affected by the validity of (5.4.36).

To this end, we also need to strengthen (5.4.2) by requiring that

$$V_k \text{ is differentiable in } B_1(0) \text{ where } |\nabla V_k| \leq A. \tag{5.4.37}$$

Proposition 5.4.27 *In addition to the assumptions of Proposition 5.4.19, suppose further that (5.4.37) holds and for some* $\delta_0 \in (0,1]$ *and property (5.4.36) is satisfied. Then (5.4.27) holds.*

The proof of Proposition 5.4.27, as well as other interesting concentration/compactness properties, are a consequence of the following result established in [BT2].

Theorem 5.4.28 *Let* u_k *satisfy (5.4.1), and assume (5.4.2), (5.4.3), (5.4.5), (5.4.36), and (5.4.37). If zero is a blow-up point for* u_k*, then there exists* $r_0 \in (0,1]$ *such that along a subsequence*

$$|z|^{2\alpha_k} V_k e^{u_k} \rightharpoonup 8\pi(1+\alpha)\delta_{z=0},$$

weakly in the sense of measure in B_{r_0}.

Before giving the proof of Theorem 5.4.28, we shall derive some of its interesting consequences.

Proof of Proposition 5.4.27. Again, we only have to consider the case where:

$$\int_{\{|z|\leq\delta_0\}} |z|^{2\alpha_k} V_k e^{u_k} \leq C.$$

Therefore, we see that u_k satisfies all assumptions of Theorem 5.4.28, and so we find $r_0 \in (0,1)$, such that along a subsequence

$$\int_{B_r} |z|^{2\alpha_k} V_k(z) e^{u_k} \to 8\pi(1+\alpha),$$

$\forall r \in (0, r_0)$, and (5.4.27) follows. □

Proposition 5.4.29 *Let u_k satisfy (5.4.1). Assume (5.4.2), (5.4.3) hold and for $\alpha > 0$ in (5.4.3), also assume (5.4.37) holds. If $u_k \geq -M$ in $B_1(0)$, then u_k is uniformly bounded in $L^\infty_{loc}(B_1(0))$.*

Proof. By replacing u_k with $u_k + M$, we can always assume that $u_k \geq 0$ in $B_1(0)$.

Claim: For every $\Omega \subset\subset B_1(0)$, there exists a constant C (depending on Ω) such that

$$\int_\Omega |z|^{2\alpha_k} V_k e^{u_k} \leq C. \tag{5.4.38}$$

To establish (5.4.38), we follow an argument given by Brézis–Merle in [BM]. Let φ_1 be the first positive eigenfunction of $-\Delta$ in $H_0^1(B_1(0))$, and denote by λ_1 the corresponding eigenvalue. We normalize φ_1 to have $\int_{B_1} \varphi_1 = 1$. We multiply equation (5.4.1) by φ_1 and integrate over B_1 to obtain:

$$\int_{B_1} |z|^{2\alpha_k} V_k e^{u_k} \varphi_1 = \lambda_1 \int_{B_1} u_k \varphi_1 + \int_{\partial B_1} u_k \frac{\partial \varphi_1}{\partial \nu} \leq \lambda_1 \int_{B_1} u_k \varphi_1. \tag{5.4.39}$$

On the other hand, by (5.4.2) and with the help of Jensen's inequality (2.5.8), we have

$$\int_{B_1} |z|^{2\alpha_k} V_k e^{u_k} \varphi_1 \geq b_1 \int_{B_1} e^{u_k + 2\alpha_k \log|z|} \varphi_1 \geq b_1 e^{\int_{B_1} u_k \varphi_1 + 2\alpha_k \int_{B_1} \log|z| \varphi_1}.$$

Thus, from (5.4.39) we derive

$$e^{\int_{B_1} u_k \varphi_1} \leq \frac{1}{b_1} \lambda_1 e^{2\alpha_k \int_{B_1} \log \frac{1}{|z|} \varphi_1} \int_{B_1} u_k \varphi_1 \leq C \int_{B_1} u_k \varphi_1,$$

which implies

$$\int_{B_1(0)} u_k \varphi_1 \leq 2C. \tag{5.4.40}$$

Inserting (5.4.40) into (5.4.39), we arrive at (5.4.38).

Now, let us argue by contradiction and assume that u_k admits a blow-up point z_0 in B_1. As a consequence of (5.4.38), we find $\delta_0 > 0$ sufficiently small, so that z_0 is the only blow-up point for u_k in $\overline{B_{\delta_0}(z_0)} \subset B_1$. So u_k is uniformly bounded in $C^{2,\gamma}_{\text{loc}}(B_{\delta_0}(z_0) \setminus \{z_0\})$, and we can pass to a subsequence to derive:

$$|z|^{2\alpha_k} V_k e^{u_k} \rightharpoonup \nu, \text{ weakly in the sense of measure in } B_{\delta_0}(z_0);$$
$$u_k \rightharpoonup \xi, \text{ uniformly in } C^2_{\text{loc}}(B_{\delta_0}(z_0) \setminus \{z_0\});$$

and $-\Delta \xi = \nu$ in the sense of distributions in $B_{\delta_0}(z_0)$.

In view of Corollary 5.4.18 (ii), $\nu(z_0) \geq 4\pi$ necessarily. This leads to the estimate:

$$\xi(z) \geq 2\log\frac{1}{|z - z_0|} - C, \qquad \text{in } B_{\delta_0}(z_0). \tag{5.4.41}$$

If $z_0 \neq 0$, it suffices to have a contradiction. In fact, in this case, we know from (5.4.38) and Fatou's lemma that e^{ξ} is integrable in $B_\delta(z_0)$; but is impossible by virtue of estimate (5.4.41).

Note that when $\alpha = 0$ in (5.4.3), the argument above leads to a contradiction also in case $z_0 = 0$.

Hence, suppose $\alpha > 0$ and $z_0 = 0$. In this situation, Fatou's lemma implies

$$\int_{B_\delta(0)} |z|^{2\alpha} e^{\xi} < +\infty, \tag{5.4.42}$$

$\forall \delta \in (0, \delta_0)$. On the other hand, since $u_k \geq 0$ and zero is the only blow-up point for u_k in $\bar{B}_{\delta_0}$, we may check that (5.4.36) holds. So we are in a position to apply Theorem 5.4.28 and conclude that $\nu(0) \geq 8\pi(1 + \alpha)$. Consequently,

$$\xi(z) \geq 4(\alpha + 1)\log\frac{1}{|z|} - C \text{ in } B_{\delta_0},$$

which clearly contradicts (5.4.42). □

Remark 5.4.30 By direct inspection of the proof above, we see that when $\alpha \in (0, 1]$, condition (5.4.37) can be weakened to (5.4.7) (see Proposition 5.4.19).

Corollary 5.4.31 *Under the assumptions of Proposition 5.4.29, if u_k blows up in $B_1(0)$ then $\inf_{B_1} u_k \to -\infty$, as $k \to +\infty$.*

Again observe that, if we know a priori that zero is *not* a blow-up point, then the conclusion of Corollary 5.4.31 follows without requiring (5.4.37).

Proposition 5.4.32 *Let u_k satisfy (5.4.1) and suppose that (5.4.2), (5.4.3), and (5.4.5) hold. In addition, assume (5.4.37) if and only if $\alpha > 0$ in (5.4.3).*

There exists $r_0 \in (0, 1]$ and a subsequence of u_k (denoted in the same way), for which only one of the following alternatives hold:
(a) u_k is bounded uniformly in $L^\infty_{loc}(B_{r_0})$;
(b) $\sup_{\Omega'} u_k \to -\infty$, for every $\Omega' \subset\subset B_{r_0}$;
(c) there exists $z_k \to 0$, with $u_k(z_k) \to +\infty$, such that
i. $\sup_{\Omega'} u_k \to -\infty$, $\forall\, \Omega' \subset\subset B_{r_0} \setminus \{0\}$,
ii. $|z|^{2\alpha_k} V_k e^{u_k} \rightharpoonup \beta\delta_{z=0}$ weakly in the sense of measures in B_{r_0}, and $\beta \geq 4\pi$.

Proof. Without loss of generality we can assume that (5.4.5) holds for $\delta_0 = 1$. Indeed, if this was not the case, then by means of (5.4.6) we would simply replace u_k with

$u_k(\delta_0 z) + 2(\alpha_k + 1)\log\delta_0$. We can apply Proposition 5.3.17 to u_k in B_1, and find a subsequence of u_k, which for simplicity we denote in the same way, such that

$$u_k^+ \text{ is uniformly bounded in } L^\infty_{\text{loc}}(B_1 \setminus S), \tag{5.4.43}$$

where S is the blow-up set (possibly empty) of (the subsequence) u_k.

We can also assume that

$$|z|^{2\alpha_k} V_k e^{u_k} \rightharpoonup \nu, \text{ weakly in the sense of measures in } B_1, \tag{5.4.44}$$

with ν a finite measure in B_1.

Note that, in view of (5.4.43), any other subsequence of u_k admits the *same* blow-up set S.

For the case when the blow-up set S is empty, u_k is uniformly bounded from above in any subset of B_1. Then we can use Proposition 5.4.29 together with Corollary 5.2.9 to conclude that, along a possible subsequence, either alternative (a) or (b) holds.

Suppose now that S is not empty, but contains a finite number of points. Then for any $\delta > 0$ sufficiently small and $z_0 \in S$, we can apply Corollary 5.4.31 to

$$\tilde{u}_k(z) = u_k(z_0 + \delta z) + 2(\alpha_k + 1)\log\delta, \; z \in B_1,$$

and see that

$$\inf_{\partial B_\delta(z_0)} u_k = \inf_{B_\delta(z_0)} u_k \to -\infty, \text{ as } k \to +\infty, \tag{5.4.45}$$

where in (5.4.45) we have used the superharmonicity of u_k. Consequently, for every $\delta > 0$ sufficiently small, there holds

$$\inf_{\Omega_{1,\delta}} u_k \to -\infty, \text{ as } k \to +\infty, \tag{5.4.46}$$

where

$$\Omega_{1,\delta} = B_1 \setminus \bigcup_{p \in S} B_\delta(p) \subset\subset B_1 \setminus S.$$

Thus, by means of (5.4.43), we can apply Corollary 5.2.9 and conclude:

$$\sup_{\Omega_{1,\delta}} u_k \to -\infty, \text{ as } k \to +\infty. \tag{5.4.47}$$

At this point the remaining part of the proof follows easily. Indeed, we can find $r_0 > 0$ such that either $B_{r_0}(0) \cap S$ is empty or $B_{r_0}(0) \cap S = \{0\}$. In the first case, by (5.4.47) we see that alternative (b) holds. While in the second case, we easily check the validity of part *i.* of property (c), and see that the measure ν in (5.4.44) is supported exactly at the origin in B_{r_0}. Namely, $\nu = \beta\delta_{z=0}$ in B_{r_0} and $\beta \geq 4\pi$ by virtue of (5.4.4). □

Remark 5.4.33 If we assume (5.4.7), then we can use Propositions 5.4.19 and 5.4.20 to deduce that part *ii.* of property (c) in Proposition 5.4.32 holds with $\beta \geq 8\pi$.

We shall present examples in Section 5.5.5 below showing that the condition $\beta \geq 8\pi$ is sharp and cannot be improved in general. This is surprising in a way, expecially if we take into account Theorem 5.4.28, where clearly the additional condition (5.4.36) must play a crucial role. The role of (5.4.36) towards the "concentration" phenomenon was pointed out first by Wolansky for the non-vanishing case, i.e., when $\alpha_k = 0$ in (5.4.1) (see also [L2]).

The general case was derived in [BT2] by means of the Pohozaev's type identity, given in (5.2.16). This approach appears particularly useful when condition (5.4.36) holds and it will be very much exploited below. See also [OS1] and [OS2] for related results.

Proof of Theorem 5.4.28. By taking a subsequence if necessary, we can assume that (5.4.44) holds, with ν a finite measure in $B_1(0)$, satisfying:

$$\nu(0) = \beta \geq 8\pi.$$

Furthermore, there exists $r_0 \in (0, \delta_0]$, such that zero is the only point of blow-up for u_k in B_{r_0}. As a consequence, we find that $f_k = |z|^{2\alpha_k} V_k e^{u_k}$ is uniformly bounded in $C^0_{\mathrm{loc}}(B_{r_0} \setminus \{0\})$. Therefore, in view of assumption (5.4.36), we can use Green's representation formula for

$$\varphi_k = u_k - \inf_{\partial B_{\delta_0}} u_k,$$

to obtain that

$$\varphi_k \to \varphi = \frac{\beta}{2\pi} \log \frac{1}{|z|} + \phi, \text{ uniformly in } C^2_{\mathrm{loc}}(B_{r_0} \setminus \{0\}), \tag{5.4.48}$$

with ϕ a regular function in B_{r_0}.

Note in particular that

$$\nabla u_k = \nabla \varphi_k \to \nabla \varphi = \frac{\beta}{2\pi} \frac{z}{|z|^2} + \nabla \phi, \tag{5.4.49}$$

uniformly in $C^1_{\mathrm{loc}}(B_{r_0} \setminus \{0\})$.

Fix $r \in (0, r_0)$, and use Pohozaev's identity (5.2.16) for u_k in B_r to obtain:

$$\begin{aligned} &r \int_{\partial B_r} \left(\frac{|\nabla u_k|^2}{2} - (\nu \cdot \nabla u_k)^2 \right) d\sigma \\ &= r \int_{\partial B_r} |z|^{2\alpha_k} V_k(z) e^{u_k} d\sigma - 2(\alpha_k + 1) \int_{B_r} |z|^{2\alpha_k} V_k e^{u_k} - \int_{B_r} (z \cdot \nabla V_k) |z|^{2\alpha_k} e^{u_k}. \end{aligned} \tag{5.4.50}$$

In view of our assumptions on V_k, we easily check that

$$\left| \int_{B_r} (z \cdot \nabla V_k) |z|^{2\alpha_k} e^{u_k} \right| \leq Cr,$$

with a suitable constant $C > 0$ independent of $k \in \mathbb{N}$ and of r.

If we pass to the limit in (5.4.50) and as $k \to +\infty$, use (5.4.49), we find

$$\lim_{k\to+\infty} r \int_{\partial B_r} |z|^{2\alpha_k} V_k e^{u_k} = -\frac{\beta^2}{4\pi} + 2(\alpha+1)\beta + o(1), \tag{5.4.51}$$

with $o(1) \to 0,$ as $r \to 0$.

Claim:

$$\inf_{\partial B_{\delta_0}} u_k \to -\infty, \text{ as } k \to +\infty. \tag{5.4.52}$$

To establish (5.4.52) we argue by contradiction and suppose that $\inf\limits_{\partial B_{\delta_0}} u_k > -M$, for $M > 0$ a suitable constant. Thus, by means of Fatou's lemma and (5.4.48), we find:

$$C > \limsup_{k\to+\infty} \int_{B_{r_0}} |z|^{2\alpha_k} e^{u_k} \geq e^{-M} \limsup_{k\to+\infty} \int_{B_{r_0}} |z|^{2\alpha_k} e^{\varphi_k} \geq e^{-M} \int_{B_{r_0}} |z|^{2\alpha-\beta/2\pi} e^{\phi}.$$

This implies

$$\beta < 4\pi(1+\alpha). \tag{5.4.53}$$

Consequently,

$$\begin{aligned} \left| \limsup_{k\to+\infty} r \int_{\partial B_r} |z|^{2\alpha_k} V_k(z) e^{u_k} d\sigma \right| &\leq e^{-M} r \int_{\partial B_r} |z|^{2\alpha} V e^{\varphi} d\sigma \\ &\leq C r^{2(\alpha+1)-\beta/2\pi} \to 0, \text{ as } r \to 0. \end{aligned} \tag{5.4.54}$$

Using (5.4.54) together with (5.4.51), as $r \to 0$, we derive

$$-\frac{\beta^2}{4\pi} + 2(\alpha+1)\beta = 0, \quad \text{i.e.,} \quad \beta = 8\pi(1+\alpha), \tag{5.4.55}$$

in contradiction with (5.4.53).

Once (5.4.52) is established, we can use (5.4.48) to conclude that, for every compact set $K \subset B_{r_0} \setminus \{0\}$,

$$\sup_K u_k \to -\infty.$$

That is,

$$|z|^{2\alpha_k} V_k e^{u_k} \to 0, \text{ uniformly in } C^0_{\text{loc}}\,(B_{r_0} \setminus \{0\}) \tag{5.4.56}$$

and

$$\nu = \beta \delta_{z=0}. \tag{5.4.57}$$

Furthermore, (5.4.56) implies that the left-hand side of (5.4.51) must vanish. So as $r \to 0$, we deduce that $\beta = 8\pi(1+\alpha)$ and we arrive at the desired conclusion. □

5.4.3 A global concentration-compactness result

In concluding our discussion on the concentration-compactenss phenomenon, we show how to patch together the "local" information derived above to obtain a "global" concentration-compactness result as follows.

Motivated by problems (5.1.1) and (5.1.2), we consider $\Omega \subset \mathbb{R}^2$ bounded open set and we let

$$W_k(z) = \prod_{i=1}^{m} |z - z_i|^{2\alpha_{i,k}} V_k(z), \; z \in \Omega, \tag{5.4.58}$$

with $\{z_1, \ldots, z_m\} \subset \Omega$ given distinct points;

$$\alpha_{i,k} \to \alpha_i \geq 0, \; i = 1, \ldots, m; \tag{5.4.59}$$

$$0 < b_1 \leq V_k(z) \leq b_2, \; \forall z \in \Omega. \tag{5.4.60}$$

We consider a sequence u_k satisfying

$$\begin{cases} -\Delta u_k = W_k e^{u_k} \text{ in } \Omega, \\ \int_\Omega W_k e^{u_k} \leq C, \end{cases} \tag{5.4.61}$$

with $C > 0$ a suitable constant.

We have:

Theorem 5.4.34 *Let u_k satisfy (5.4.61) and assume (5.4.58), (5.4.59), and (5.4.60) hold. In addition, if (5.4.58) holds with $\alpha_j > 0$ for some $j \in \{1, \ldots, m\}$, suppose that: $V_k \in C^{0,1}(B_{r_0}(z_j))$ and $|\nabla V_k| \leq A$ in $B_{r_0}(z_j)$, for suitable $r_0 > 0$ sufficiently small and $A > 0$. Along a subsequence, one of the following alternative holds:*
(a) u_k is uniformly bounded in $L^\infty_{loc}(\Omega)$;
(b) $\sup_{\Omega'} u_k \to -\infty$, as $k \to \infty$, for every $\Omega' \subset\subset \Omega$;
(c) there exists a finite set $S = \{q_1, \ldots, q_s\} \subset \Omega$ (a blow-up set) and corresponding sequences $\{z_{j,k}\} \subset \Omega$ such that as $k \to \infty$
(i.) $z_{j,k} \to q_j$ and $u_k(z_{j,k}) \to +\infty$, $\forall j = 1, \ldots, s$,
(ii.)

$$\sup_{\Omega'} u_k \to -\infty, \; \textit{for every } \Omega' \subset\subset \Omega \setminus S, \tag{5.4.62}$$

(iii.)

$$W_k u_k \to \sum_{j=1}^{s} \beta_j \delta_{q_j}, \tag{5.4.63}$$

weakly in the sense of measure in Ω, with $\beta_j \geq 4\pi$, for every $j = 1, \ldots, s$.
In addition, if $V_k \to V$ in $C^0_{loc}(\Omega)$ then (5.4.63) holds with $\beta_j \geq 8\pi$, $j = 1, \ldots, s$.

Proof. Clearly, we can apply Proposition 5.3.17 to find a subsequence of u_k, denoted in the same way, such that

$$u_k^+ \text{ is uniformly bounded in } L^\infty_{\text{loc}}(\Omega \setminus S), \tag{5.4.64}$$

where S is the (possibly empty) blow-up set of the subsequence of u_k. We also may suppose that as $k \to \infty$,

$$W_k e^{u_k} \to \nu, \text{ weakly in the sense of measure in } \Omega, \tag{5.4.65}$$

where ν is a finite measure in Ω.

As in the proof of Proposition 5.4.32, we see that either u_k satisfies alternative (a), or : $\sup_{\Omega'} u_k \to -\infty$, as $k \to \infty$, for every $\Omega' \subset\subset \Omega \setminus S$.

In particular, if S is empty then either alternative (a) or (b) holds. On the other hand, when S is not empty, it contains a finite number of points, say $S = \{q_1, \dots, q_s\}$, for which $(c)(i)$ and $(c)(ii)$ hold. Hence, the measure ν in (5.4.65) is supported in S. Now, for $\delta > 0$ sufficiently small and $q \in S$, we can apply Proposition 5.4.32 to $\tilde{u}_k = u_k(q + \delta z) + 2(\alpha_k + 1)\log\delta,\ z \in B_1$; and we conclude that

$$\nu|_{B_\delta(q)} = \beta\delta_q \text{ with } \beta \geq 4\pi.$$

Thus, $(c)(iii)$ is established once we also take into account Remark 5.4.33. □

5.5 A quantization property in the concentration phenomenon

5.5.1 Preliminaries

The goal of this section is to give a precise characterization of the concentration value β within that occurs in alternative (c) of Proposition 5.4.32.

For this purpose, we take u_k to satisfy

$$-\Delta u_k = |z|^{2\alpha_k} V_k e^{u_k} \quad \text{in } B_1, \tag{5.5.1}$$

$$|z|^{2\alpha_k} V_k e^{u_k} \rightharpoonup \beta\delta_{z=0}, \tag{5.5.2}$$

weakly in the sense of measure in B_1.

It follows from Theorem 5.4.28 that, when u_k satisfies also (5.4.36), then necessarily

$$\beta = 8\pi(1 + \alpha), \tag{5.5.3}$$

provided that (5.4.2), (5.4.3) and (5.4.37) also hold.

Explicit examples discussed in Section 5.5.5 below show that when (5.4.36) is not satisfied, then (5.5.3) fails to hold in general.

On the other hand, such examples also indicate that in any case β cannot take any arbitrary value larger than or equal to 8π, but in fact must be restricted to satisfying a sort of "quantization" property as follows:

$$\beta \in 8\pi\mathbb{N} \cup 8\pi(\mathbb{N} + \alpha). \tag{5.5.4}$$

It is one of the main goals of this section to prove (5.5.4) and thus complete Theorem 5.4.34 as follows:

Theorem 5.5.35 *If alternative (c) holds in Theorem 5.4.34, then property (iii) is verified with,*

$$\beta_j \in 8\pi\mathbb{N},\ \textit{for}\ q_j \notin \{z_1,\ldots,z_m\}\ \textit{or}\ \beta_j \in \{8\pi(\mathbb{N}+\alpha_i)\} \cup 8\pi\mathbb{N}\ \textit{for}\ q_j = z_i, \tag{5.5.5}$$

for some $i = 1,\ldots,m$ *and* $j = 1,\ldots,s$.

If we take $\alpha_{i,k} = 0$ in (5.4.58), then (5.5.5) gives $\beta_j \in 8\pi\mathbb{N}$, $\forall\ j = 1,\ldots,s$. This situation was handled first by Li–Shafrir in [LS], while the general case was established by the author in [T5].

Theorem 5.5.35 easily follows once (5.5.4) is established. In fact, we can localize our analysis around each blow-up point q_j, and after a suitable translation, we can scale our sequence according to (5.2.13) to obtain a problem of the type (5.5.1) and (5.5.2), for which whenever $q_j \notin \{z_1,\ldots,z_m\}$, we take $\alpha_k = 0\ \forall\ k$.

Let us mention that in the process of establishing (5.5.4), we also obtain an inequality of the type "sup + inf" in the same spirit of [BLS], [ChL4], and [Sh]. This will be discussed in Section 5.5.3.

5.5.2 A version of Harnack's inequality

Throughout this section we assume

$$\alpha_k \geq 0 \text{ and } \alpha_k \to \alpha, \tag{5.5.6}$$

and

$$V_k \in C^{0,1}(B_1):\ 0 < b_1 \leq V_k \leq b_2,\ |\nabla V_k| \leq A \text{ in } B_1. \tag{5.5.7}$$

The main technical ingredient to derive (5.5.4) is contained in the following:

Theorem 5.5.36 *Let* u_k *satisfy (5.5.1), and assume (5.5.6) and (5.5.7) hold. Suppose there exists* $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$, $C_0 > 0$, *and a sequence* $\{z_k\} \subset B_1$ *such that*

(i)

$$z_k \to 0, \quad u_k(z_k) + 2(\alpha_k + 1)\log|z_k| \to +\infty; \tag{5.5.8}$$

(ii)

$$\sup_{|z|\leq 2\varepsilon_0|z_k|} \{u_k(z) + 2(\alpha_k+1)\log|z|\} \leq C_0; \tag{5.5.9}$$

(iii)

$$\int_{|z|\leq(1+\varepsilon_0)|z_k|} |z|^{2\alpha_k} V_k e^{u_k} \leq C_0. \tag{5.5.10}$$

Let

$$v_k(z) = u_k(|z_k|z) + 2(\alpha_k + 1)\log|z_k|, \tag{5.5.11}$$

then along a subsequence, the following alternative holds:

(a) either $\max_{\{|z|\leq\varepsilon_0\}} v_k \to -\infty$ *and* $\inf_{B_1} u_k \leq \max_{\{|z|\leq\varepsilon_0\}} v_k + 2(\alpha_k + 1)\log|z_k| + C$*,*

(b) or $v_k(0) \to +\infty$ *and* $\inf_{B_1} u_k \leq -u_k(0) + C$*,*

for a suitable constant C depending only on b_1, b_2*, and A in (5.5.7).*

Proof. To simplify our notation, we take $\alpha_k = \alpha, \forall k \in \mathbb{N}$. Furthermore, by passing to a subsequence if necessary, from (5.5.7), we can further ensure that

$$V_k \to V, \quad \text{uniformly in} \quad C^0_{\text{loc}}(B_1), \tag{5.5.12}$$

where again, we loose no generality by assuming that

$$V(0) = 1. \tag{5.5.13}$$

Observe that v_k satisfies:

$$\begin{cases} -\Delta v_k = |z|^{2\alpha} V_k(|z_k|z)e^{v_k} \text{ in } D = \{|z| \leq 1 + \varepsilon_0\}, \\ \int_D |z|^{2\alpha} V_k(|z_k|z)e^{v_k} \leq C_0, \\ \sup_{|z|\leq\varepsilon_0} \{v_k(z) + 2(\alpha + 1)\log|z|\} \leq C_0. \end{cases} \tag{5.5.14}$$

Thus in view of Corollary 5.4.24, along a subsequence, we see that either

$$v_k(0) = \max_{|z|\leq\varepsilon_0} v_k + O(1) \to \infty, \text{ as } k \to +\infty, \tag{5.5.15}$$

or

$$\max_{|z|\leq\varepsilon_0} v_k < C.$$

In the latter case, we can derive a stronger statement, by taking into account that

$$v_k\left(\frac{z_k}{|z_k|}\right) \to +\infty \text{ as } k \to +\infty; \tag{5.5.16}$$

that is, (a subsequence of) v_k admits a blow-up point on the unit circle.

Therefore, v_k verifies alternative (c) in Theorem 5.5.35, and by means of (5.4.62) we conclude:

$$\max_{|z|\leq\varepsilon_0} v_k \to -\infty, \text{ as } k \to +\infty. \tag{5.5.17}$$

In order to proceed further, we observe the following facts.
Fact 1: If (5.5.15) holds, set

$$\varepsilon_k = e^{-\frac{u_k(0)}{2(\alpha+1)}} \to 0. \tag{5.5.18}$$

Then along a subsequence,

$$\xi_k(z) := u_k(\varepsilon_k z) + 2(\alpha + 1)\log \varepsilon_k \tag{5.5.19}$$

satisfies

$$\xi_k(z) \to \xi \quad \text{uniformly in } C^2_{\text{loc}}(\mathbb{R}^2), \tag{5.5.20}$$

with ξ defined in (5.4.12) and (5.4.14) for $\alpha = a$ and $\mu = 1$. More precisely,

$$\xi(z) = \log \frac{\lambda_0}{(1 + \lambda_0 \gamma_\alpha |z^{\alpha+1} - y_0|^2)^2},\ \gamma_\alpha = \frac{1}{8(\alpha + 1)^2}, \tag{5.5.21}$$

where $\lambda_0 = 1$ and $y_0 = 0$ for $\alpha \in (0, +\infty) \setminus \mathbb{N}$ while for $\alpha \in \mathbb{N}$, we have;

$$s_0 = \frac{1}{\gamma_\alpha}|y_0| \leq \frac{1}{4} \text{ and } \lambda_0 = \frac{1}{2s_0^2}(1 - 2s_0 \pm \sqrt{1 - 4s_0}) \geq 1. \tag{5.5.22}$$

Proof of Fact 1. Let $\lambda_k = e^{\frac{v_k(0)}{2(\alpha+1)}} \to +\infty$, as $k \to +\infty$, and observe that

$$\xi_k(z) = v_k\left(\frac{z}{\lambda_k}\right) + 2(\alpha + 1)\log \frac{1}{\lambda_k}.$$

Thus, by setting $R_k = \varepsilon_0 \lambda_k$, from (5.5.15) we see that,

$$\max_{\{|z| \leq R_k\}} \xi_k = \max_{\{|z| \leq \varepsilon_0\}} v_k - v_k(0) = O(1), \text{ as } k \to +\infty.$$

Note also that $\xi_k(0) = 0$.

At this point, we can use (5.5.14) together with (5.5.12) and (5.5.13) to check that ξ_k satisfies all assumptions of Lemma 5.4.21 with $y_k = 0$, $R_k = \varepsilon_0\lambda_k \to +\infty$, and

$$U_k(z) = |z|^{2\alpha} V_k\left(\frac{|z_k|}{\lambda_k} z\right) \to |z|^{2\alpha} \text{ in } C^0_{\text{loc}}(\mathbb{R}^2).$$

Thus, we readily deduce (5.5.20), while (5.5.21) follows from Remak 5.4.22.

In case (5.5.15) is not available, we can confide in (5.5.16) to obtain an analogous result. To this purpose, suppose that along a subsequence,

$$\frac{z_k}{|z_k|} \to z_0 \text{ as } k \to +\infty, \quad \text{with} \quad |z_0| = 1. \tag{5.5.23}$$

Hence z_0 is a blow-up point for v_k, and we find $r_0 \in (0, \varepsilon_0)$ such that z_0 is the *only* blow-up point for v_k in $\overline{B_{r_0}(z_0)}$. Let $y_k \in \overline{B_{r_0}(z_0)}$:

$$v_k(y_k) = \max_{B_{r_0}(z_0)} v_k. \tag{5.5.24}$$

Hence, along a possible subsequence:

$$y_k \to z_0 \text{ and } v_k(y_k) \to +\infty.$$

Fact 2: Set

$$\delta_k = e^{-\frac{1}{2}v_k(y_k)} \to 0, \quad \xi_k(z) = v_k(y_k + \delta_k y) + 2\log\delta_k \tag{5.5.25}$$

(along a subsequence) we have

$$\xi_k \to \xi = \log\left(\frac{1}{\left(1+\frac{1}{8}|z|^2\right)^2}\right), \text{ uniformly in } C^2_{\text{loc}}(\mathbb{R}^2) \tag{5.5.26}$$

Proof of Fact 2. For $R_k = \frac{r_0}{\delta_k} \to +\infty$, observe that

$$\begin{cases} -\Delta\xi_k = U_k e^{\xi_k} \quad \text{in } D_k = \{|z| \le R_k\}, \\ \max\limits_{\bar{D}_k} \xi_k = 0 = \xi_k(0), \\ \int\limits_{D_k} U_k e^{\xi_k} \le C_0, \end{cases}$$

with $U_k(z) = |y_k + \delta_k z|^{2\alpha} V_k(|z_k|y_k + |z_k|\delta_k z) \to 1$, uniformly in $C^0_{\text{loc}}(\mathbb{R}^2)$. So we are now in position to apply Lemma 5.4.21 with $a = 0$, $z_0 = 0$ and $\mu = 1$. As above we arrive at the desired conclusion by virtue of Remark 5.4.22, where the additional information, $\xi(0) = \max\limits_{\mathbb{R}^2} \xi = 0$, implies that we must have $\lambda_0 = 1$ and $y_0 = 0$ in (5.4.13). □

To proceed further, we use a moving-plane technique as introduced in the same context by Brezis–Li–Shafrir in [BLS]. To this purpose, we assume, without loss of generality, that u_k is defined up to the boundary ∂B_1, otherwise for any fixed $r_0 \in (0, 1)$, we simply need to replace u_k with $u_k(r_0 z) + 2(\alpha + 1)\log r_0$.

Define

$$\omega_k(t,\theta) = u_k(e^{t+i\theta}) + 2(\alpha+1)t - \frac{A}{b_1}e^t, \tag{5.5.27}$$

for $(t,\theta) \in Q = (-\infty, 0] \times [0, 2\pi)$.

A simple calculation shows that

$$-\Delta\omega_k = \tilde{V}_k(t,\theta)e^{\omega_k} + \frac{A}{b_1}e^t \text{ in } Q,$$

with $\tilde{V}_k(t,\theta) = V_k(e^{t+i\theta})e^{\frac{A}{b_1}e^t}$.

Thus, we can use assumption (5.5.7) to obtain:

$$\frac{\partial}{\partial t}\left(\tilde{V}_k(t,\theta)e^{\xi}+\frac{A}{b_1}e^t\right)\geq 0, \quad \forall\,\xi\in\mathbb{R} \text{ and } (t,\theta)\in Q. \tag{5.5.28}$$

Claim 1: For k fixed, there exists $\lambda < 0$ (depending on k) such that for every $\mu \leq \lambda$ there holds:

$$\omega_k(2\mu-t,\theta)-\omega_k(t,\theta)<0, \text{ for } \mu<t<0 \text{ and } \theta\in[0,2\pi). \tag{5.5.29}$$

To establish Claim 1 observe that

$$\omega_k(2\mu-t,\theta)-\omega_k(t,\theta)\leq\mu+c_k, \forall t\in\left[\frac{\mu}{2},0\right) \text{ and } \theta\in[0,2\pi);$$

while

$$\frac{\partial}{\partial t}\omega_k(t,\theta)\geq 2(\alpha+1)-c_ke^{\mu/2} \quad \forall t<\mu/2 \text{ and } \theta\in[0,2\pi),$$

for suitable $c_k > 0$ depending only on k. Thus, we can choose λ sufficiently negative (depending on k) such that $\forall\mu\leq\lambda$,

$$\omega_k(2\mu-t,\theta)-\omega_k(t,\theta)<0, \text{ for } t\in\left[\frac{\mu}{2},0\right) \text{ and } \theta\in[0,2\pi);$$

$$\frac{\partial}{\partial t}\omega_k(t,\theta)>0, \quad \text{for } t<\frac{\mu}{2} \text{ and } \theta\in[0,2\pi);$$

and this ensures (5.5.29).

Therefore, the following is well-defined:

$$\lambda_k=\sup\{\lambda\leq 0 : (5.5.29) \text{ holds, for every } \mu\leq\lambda\}. \tag{5.5.30}$$

Claim 2:

$$\min_{\theta\in[0,2\pi)}\omega_k(0,\theta)\leq\max_{\theta\in[0,2\pi)}\omega_k(2\lambda_k,\theta) \tag{5.5.31}$$

To obtain (5.5.31) we consider the function $\psi_k(t,\theta)=\omega_k(2\lambda_k-t,\theta)-\omega_k(t,\theta)$. Hence, $\psi_k\leq 0$, and by virtue of (5.5.28), we see that $\Delta\psi_k\geq 0$, for $(t,\theta)\in[\lambda_k,0]\times[0,2\pi)$. Thus, we can use the maximality of λ_k together with the maximum principle to conclude that necessarily ψ_k must vanish in $\{0\}\times[0,2\pi)$. This fact readily implies (5.5.31).

Our next task is to estimate λ_k. We must proceed differently according to whether alternative (a) or (b) holds.

We start by analyzing alternative (a) relative to the validity of the following:

Lemma 5.5.37 *If (5.5.18), (5.5.20), and (5.5.21) are verified, then in (5.5.30), we have*

$$\lambda_k\leq\log\varepsilon_k+O(1), \tag{5.5.32}$$

and

$$\inf_{B_1}u_k\leq -u_k(0)+O(1), \tag{5.5.33}$$

as $k\to+\infty$.

Proof of Lemma 5.5.37. For $(t, \theta) \in Q$, define

$$\omega(t,\theta) = \xi(e^{t+i\theta}) + 2(\alpha+1)t = \log \frac{\lambda_0 e^{2(\alpha+1)t}}{\left(1 + \lambda_0 \gamma_\alpha |e^{(\alpha+1)(t+i\theta)} - y_0|^2\right)^2},$$

with $\gamma_\alpha = \frac{1}{8(\alpha+1)^2}$, $y_0 \in \mathbb{C}$ and $\lambda_0 > 0$ specified according to (5.5.22).

By means of Proposition 2.2.3, we see that for fixed $\theta \in [0, 2\pi)$, the function $\omega(\cdot, \theta)$ is symmetric with respect to the axis $t = \log \frac{1}{\sqrt{\tau}}$, with $\tau = \left(\lambda_0 \gamma_\alpha^2\right)^{\frac{1}{2(\alpha+1)}}$. Namely, $\omega\left(\log \frac{1}{\sqrt{\tau}} - t, \theta\right) = \omega\left(\log \frac{1}{\sqrt{\tau}} + t, \theta\right)$, $\forall t \in \mathbb{R}, \forall \theta \in [0, 2\pi)$. Moreover $\omega(\cdot, \theta)$ is increasing for $t < \log \frac{1}{\sqrt{\tau}}$, decreasing for $t > \log \frac{1}{\sqrt{\tau}}$ and attains its strict maximum value at $t = \log \frac{1}{\sqrt{\tau}}$. Notice also that

$$\omega(t,\theta) \le 2(\alpha+1)t + \log \lambda_0. \tag{5.5.34}$$

In view of (5.5.20) and (5.5.21), for every fixed $s \in \mathbb{R}$, we have

$$\sup_{\{t \le s, \theta \in [0,2\pi)\}} |\omega_k(t + \log \varepsilon_k, \theta) - \omega(t,\theta)| \to 0, \text{ as } k \to +\infty. \tag{5.5.35}$$

Thus for large k, we deduce

$$\sup_{\{t \le 4 + \log \frac{1}{\sqrt{\tau}}, \theta \in [0,2\pi)\}} |\omega_k(t + \log \varepsilon_k, \theta) - \omega(t,\theta)| < 1, \tag{5.5.36}$$

and

$$\omega_k\left(4 + \log \frac{1}{\sqrt{\tau}} + \log \varepsilon_k, \theta\right) < \omega_k\left(\log \frac{1}{\sqrt{\tau}} + \log \varepsilon_k, \theta\right), \forall \theta \in [0, 2\pi). \tag{5.5.37}$$

As a consequence of (5.5.37) we check that (5.5.29) fails to hold when $\lambda = \log \varepsilon_k + \log \frac{1}{\sqrt{\tau}} + 2$, $t = \log \varepsilon_k + \log \frac{1}{\sqrt{\tau}} + 4$, and k is large. From this fact, we immediately deduce the estimate (5.5.32).

Furthermore, using (5.5.36) and (5.5.34), for k large we can estimate:

$$\begin{aligned} \omega_k(2\lambda_k, \theta) &\le \omega(2\lambda_k - \log \varepsilon_k, \theta) + 1 \le 2(\alpha+1)(2\lambda_k - \log \varepsilon_k) + O(1) \\ &\le 2(\alpha+1)\log \varepsilon_k + O(1) = -u_k(0) + O(1). \end{aligned}$$

Therefore, using (5.5.31) and (5.5.27), we find

$$\begin{aligned} \inf_{B_1} u_k = \min_{\partial B_1} u_k &\le \min_{\theta \in [0,2\pi)} \omega_k(0,\theta) \le \max_{\theta \in [0,2\pi)} \omega_k(2\lambda_k, \theta), \\ &\le -u_k(0) + O(1), \end{aligned}$$

as $k \to +\infty$, and (5.5.33) follows. □

Conclusion of the proof of Theorem 5.5.36. Note that by virtue of Fact 1 and Lemma 5.5.37, the statement in alternative (b) is established. Concerning alternative (a) we have:

Claim 4: If (5.5.17) holds, then

$$\lambda_k \le \log |z_k| + O(1), \text{ as } k \to +\infty. \tag{5.5.38}$$

To establish (5.5.38), notice that from (5.5.25) and (5.5.26) we find a suitable $\sigma > 0$, such that for k sufficiently large, we have

$$v(y_k + \delta_k z) \le v_k(y_k) - 2\sigma, \ \frac{1}{2} \le |z| \le 3, \tag{5.5.39}$$

with y_k defined in (5.5.24). Let $\rho_k \in (0, +\infty)$ and $\theta_k \in [0, 2\pi)$ be the polar coordinates for y_k; that is,

$$\rho_k e^{i\theta_k} = y_k,$$

so $\rho_k \to 1$, as $k \to +\infty$. Since

$$\begin{aligned}\omega_k \left(\log \left((1+s)^2 \rho_k |z_k|\right), \theta_k\right) =& v_k \left((1+s)^2 y_k\right) + 2(\alpha+1) \log \left(\rho_k (1+s)^2\right) \\ & - \frac{A}{b_1} |z_k| \rho_k (1+s)^2,\end{aligned}$$

$\forall\, s > 0$, we can use (5.5.39) to deduce

$$\omega_k (\log |z_k| + \log \rho_k + 2 \log(1+\delta_k), \theta_k) < \omega_k (\log |z_k| + \log \rho_k, \theta_k) - \sigma, \tag{5.5.40}$$

provided that k is sufficiently large.

This shows that, for k sufficiently large, $\lambda = \log |z_k| + \log \rho_k + \log(1+\delta_k)$ and $t = \log |z_k| + \log \rho_k + 2 \log(1+\delta_k)$ results in the failure of inequality (5.5.29) to hold for $\theta = \theta_k$. We thus conclude (5.5.38).

At this point, we are ready to derive part (a) of our statement. Indeed, from (5.5.31) we have

$$\begin{aligned}\inf_{B_1} u_k &= \inf_{\partial B_1} u_k = \min_{\theta \in [0,2\pi)} \omega_k(0, \theta) + A/b_1 \le \max_{\theta \in [0,2\pi]} \omega_k(2\lambda_k, \theta) + A/b_1 \\ &= \max_{\theta \in [0,2\pi)} v_k \left(\frac{e^{2\lambda_k + i\theta}}{|z_k|} \right) + 2(\alpha+1)(2\lambda_k - \log |z_k|) + A/b_1 \\ &\le \max_{|z| \le R_0 |z_k|} v_k + 2(\alpha+1) \log |z_k| + C,\end{aligned} \tag{5.5.41}$$

for suitable constants R_0 and C. This completes the proof of Theorem 5.5.36. □

Remark 5.5.38 Note that inequality (5.5.31) contains a slightly stronger statement for alternative (a) of Theorem 5.5.36.

5.5.3 Inf + Sup estimates

In this section we discuss an interesting consequence of Theorem 5.5.36, concerning suitable "inf+sup" estimates that are valid for solutions of the equation:

$$-\Delta u = |z|^{2\alpha} V(z) e^u \text{ in } B_1, \tag{5.5.42}$$

with V satisfying,

$$0 < b_1 \le V \le b_2, \quad |\nabla V| \le A. \tag{5.5.43}$$

Theorem 5.5.39 *Let $\alpha \geq 0$ and let u be a solution of (5.5.42) with V satisfying (5.5.43) in B_1. Then*

$$u(0) + \inf_{B_1} u \leq C, \tag{5.5.44}$$

with C a constant depending only on α, b_1, b_2 and A.

Estimate (5.5.44) was established by Brezis–Li–Shafrir in [BLS] for the case $\alpha = 0$. However, when $\alpha = 0$, the origin plays no special role in equation (5.5.42). In fact, we can use a translation to see that (5.5.44) holds when zero is replaced by another point z_0 and B_1 is replaced by $B_1(z_0)$. Consequently, for $\alpha = 0$, one can actually conclude the following "global" result:

Corollary 5.5.40 *([BLS]) Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain and $K \subset \Omega$ be a compact set. If u is a solution to*

$$-\Delta u = V e^u \quad \text{in } \Omega, \tag{5.5.45}$$

with V satisfying (5.5.43) in Ω, then

$$\max_K u + \inf_\Omega u \leq C, \tag{5.5.46}$$

with C a suitable constant depending only on b_1, b_2, A, and dist $(K, \partial\Omega)$.

Proof of Corollary 5.5.40. Let $z_0 \in K$ satisfy $u(z_0) = \max_K u$, and let $r_0 \in (0, 1]$ be such that $\overline{B_{r_0}(z_0)} \subset \Omega$. The function $\tilde{u}(z) = u(z_0 + r_0 z) + 2 \log r_0$ satisfies $-\Delta\tilde{u} = \tilde{V}(z)e^{\tilde{u}}$ in B_1 with $\tilde{V}(z) = V(z_0 + r_0 z)$. We easily verify that $\tilde{V}$ satisfies (5.5.43) in B_1. So we can apply Theorem 5.5.39 with $\alpha = 0$ and conclude:

$$\max_K u + \inf_\Omega u \leq u(z_0) + \inf_{\{|z-z_0|<r_0\}} u = \tilde{u}(0) + \inf_{B_1} \tilde{u} - 4 \log r_0 \leq C + 4 \log \frac{1}{r_0},$$

thus the desired conclusion follows. □

On the contrary, when $\alpha > 0$ and $0 \in K$ then (5.5.44) no longer suffices to provide a "global" estimate of the (5.5.46) type. In this case (5.5.44) gives a weaker statement than (5.5.46); since it is possible to construct a sequence u_k of solutions of (1.3.25), with $V_k = 1$, that admits zero as a blow-up point in B_1, and at the same time, $u_k(0) \to -\infty$, as $k \to +\infty$. We refer to Section 5.5.5 below for details.

Therefore, the validity of (5.5.46) for $\alpha > 0$ and $0 \in K$, remains a challenging *open problem*. Here we will be able to prove (5.5.46) only under some additional assumptions (see Corollary 5.5.42 and Theorem 5.6.59).

In order to establish (5.5.44), we shall need some preliminary information. We are going to argue by contradiction and assume that there exists a sequence u_k such that

$$-\Delta u_k = |z|^{2\alpha} V_k e^{u_k} \text{ in } B_1, \tag{5.5.47}$$

with V_k satisfying (5.5.7), and

$$u_k(0) + \inf_{B_1} u_k \to +\infty. \tag{5.5.48}$$

Without loss of generality, and by passing to a subsequence if necessary we can further assume that (5.5.12) and (5.5.13) hold. Note that

$$\varepsilon_k := e^{-\frac{u_k(0)}{2(\alpha+1)}} \to 0, \text{ as } k \to +\infty, \tag{5.5.49}$$

as easily follows from (5.5.48).

Lemma 5.5.41 *For a given $k \in \mathbb{N}$, there exists $r_k \in (0, 1]$ such that*

$$\int_{\{|z|\leq r_k\}} |z|^{2\alpha} V_k e^{u_k} \leq 8\pi(1+\alpha), \tag{5.5.50}$$

and

$$\frac{r_k}{\varepsilon_k} \to +\infty. \tag{5.5.51}$$

Proof. We adapt an argument of Shafrir [Sh], also used in [BLS]. Fix $k \in \mathbb{N}$. If $\int_{B_1} |z|^{2\alpha} V_k e^{u_k} \leq 8\pi(1+\alpha)$, then we just take $r_k = 1$. Hence suppose that

$$\int_{B_1} |z|^{2\alpha} V_k e^{u_k} > 8\pi(1+\alpha), \tag{5.5.52}$$

and for $r \in (0, 1)$ define

$$G(r) = u_k(0) + \frac{1}{2\pi r} \int_{\partial B_r} u_k d\sigma + 4(\alpha+1)\log r.$$

Whence

$$\begin{aligned} G'(r) &= \frac{1}{2\pi r} \int_{\partial B_r} \frac{\partial u_k}{\partial r} + \frac{4(\alpha+1)}{r} = \frac{1}{2\pi r}\left(\int_{B_r} \Delta u_k + 8\pi(1+\alpha) \right) \\ &= \frac{1}{2\pi r}\left(8\pi(1+\alpha) - \int_{B_r} |z|^{2\alpha} V_k(z) e^{u_k} \right). \end{aligned}$$

Therefore, in view of (5.5.52), there exists a unique $r_k \in (0, 1)$ such that

$$\int_{B_{r_k}} |z|^{2\alpha} V_k(z) e^{u_k} = 8\pi(1+\alpha), \tag{5.5.53}$$

and

$$G(r_k) = \max\{G(r), r \in (0, 1)\}. \tag{5.5.54}$$

Using the super harmonicity of u_k, and as a consequence of (5.5.48) and (5.5.54), we find:

$$2\left(u_k(0)+2(\alpha+1)\log r_k\right) \geq u_k(0)+\frac{1}{2\pi r_k}\int_{\partial B_{r_k}} u_k d\sigma + 4(\alpha+1)\log r_k$$
$$= G(r_k) \geq G(r) \geq u_k(0)+\inf_{\partial B_r} u_k = u_k(0)+\inf_{B_r} u_k$$
$$\geq u_k(0)+\inf_{B_1} u_k \to +\infty, \text{ as } k \to +\infty.$$

Thus $u_k(0)+2(\alpha+1)\log r_k \to +\infty$, as $k \to +\infty$, and we derive

$$\frac{r_k}{\varepsilon_k} = e^{\frac{1}{2(\alpha+1)}(u_k(0)+2(\alpha+1)\log r_k)} \to +\infty, \text{ as } k \to +\infty,$$

as claimed. □

Proof of Theorem 5.5.39. Under the assumption of contradiction by (5.5.48), set

$$u_{1,k}(z) = u_k(r_k z) + 2(\alpha+1)\log r_k, \tag{5.5.55}$$

with r_k as given in Lemma 5.5.41. Observe that

$$\begin{cases} -\Delta u_{1,k} = |z|^{2\alpha} V_k(r_k z) e^{u_{1,k}} \text{ in } B_k = \left\{ |z| \leq \frac{1}{r_k} \right\}, \\ \int_{B_k} |z|^{2\alpha} V_k(r_k z) e^{u_{1,k}} \leq 8\pi(1+\alpha), \\ u_{1,k}(0) \to +\infty, \quad \text{as } k \to +\infty. \end{cases} \tag{5.5.56}$$

Hence, according to Proposition 5.4.26, we have that the following alternative holds around the origin:
either

$$\sup_{|z|\leq 2\varepsilon_0} \{u_{1,k}(z) + 2(\alpha+1)\log|z|\} \leq C, \tag{5.5.57}$$

or

$$\exists z_{1,k} \to 0, \;\; u_{1,k}(z_{1,k}) + 2(\alpha+1)\log|z_{1,k}| \to +\infty,$$

$$\sup_{|z|\leq 2\varepsilon_0 |z_{1,k}|} \{u_{1,k}(z) + 2(\alpha+1)\log|z|\} \leq C, \tag{5.5.58}$$

for suitable $\varepsilon_0 \in (0, \frac{1}{2})$ and $C > 0$.

In case (5.5.57) holds, we use Corollary 5.4.24 to write:

$$u_{1,k}(0) = \max_{|z|\leq 2\varepsilon_0} u_{1,k} + O(1), \text{ as } k \to +\infty.$$

Consequently,

$$\xi_k(z) = u_k(\varepsilon_k z) + 2(\alpha + 1)\log \varepsilon_k = u_{1,k}\left(\frac{\varepsilon_k}{r_k} z\right) - u_{1,k}(0)$$

satisfies all assumptions of Lemma 5.4.21 for $R_k = \frac{r_k}{\varepsilon_k}\varepsilon_0 \to +\infty$, $y_k = 0$ and $U_k(z) = |z|^{2\alpha} V_k(\varepsilon_k z) \to |z|^{2\alpha}$ in $C^0_{\text{loc}}(\mathbb{R}^2)$, as $k \to +\infty$. So we conclude that (5.5.20) and (5.5.21) hold for ξ_k, and we can apply Lemma 5.5.37 to find

$$u_k(0) + \inf_{B_1} u_k \leq C,$$

in contradiction with (5.5.48). On the other hand, if (5.5.58) holds, then we are in a position to apply Theorem 5.5.36 to u_k with $z_k = r_k z_{1,k}$. Since alternative (b) immediately leads to a contradiction of (5.5.48), we suppose that $\upsilon_k(z) = u_k(|z_k| z) + 2(\alpha + 1) \log |z_k|$ satisfies

$$\max_{|z| \leq \varepsilon_0} \upsilon_k \to -\infty, \tag{5.5.59}$$

and so

$$\inf_{B_1} u_k \leq \max_{\{|z| \leq \varepsilon_0\}} \upsilon_k + 2(\alpha + 1)\log |z_k| + C. \tag{5.5.60}$$

Conditions (5.5.59) and (5.5.60) still permit us to contradict (5.5.48) as follows:

$$u_k(0) + \inf_{B_1} u_k \leq \upsilon_k(0) + \max_{\{|z| \leq \varepsilon_0\}} \upsilon_k + C \leq 2 \max_{\{|z| \leq \varepsilon_0\}} \upsilon_k + C \to -\infty,$$

as $k \to +\infty$. And we conclude the validity of (5.5.44) as desired. □

By Theorem 5.5.39, we can check the validity of (5.5.46) as follows:

Corollary 5.5.42 *For a given $c_0 > 0$, let u satisfy (5.5.42) where V satisfies (5.5.43) and*

$$\sup_{B_1}\{u(z) + 2(\alpha + 1)\log |z|\} \leq c_0.$$

For every $r \in (0, 1)$, there exists a constant $C = C(r, \alpha, b_1, b_2, A)$ such that

$$\sup_{B_r} u + \inf_{B_1} u \leq C. \tag{5.5.61}$$

Proof. As before we argue by contradiction and by virtue of Corollary 5.5.40 we suppose the existence of u_k satisfying (5.5.47) with V_k as in (5.5.7) and such that the following conditions hold:

$$\max_{\{|z| \leq 1\}} \{u_k + 2(\alpha + 1)\log |z|\} \leq c_0, \tag{5.5.62}$$

and for a suitable sequence $\{z_k\} \subset B_1$,

$$z_k \to 0 \text{ and } u_k(z_k) + \inf_{B_1} u_k \to +\infty. \tag{5.5.63}$$

Clearly $u_k(z_k) \to +\infty$, and so $\varepsilon_k = e^{-\frac{u_k(z_k)}{2(\alpha+1)}} \to 0$, while (5.5.62) implies

$$\left|\frac{z_k}{\varepsilon_k}\right| = O(1), \text{ as } k \to +\infty. \tag{5.5.64}$$

Exactly as in Lemma 5.5.41, property (5.5.63) allows one to find $r_k \in (0, 1]$ such that

$$\int\limits_{B_{r_k}(z_k)} |z|^{2\alpha} V_k e^{u_k} \le 8\pi(1+\alpha) \text{ and } \frac{r_k}{\varepsilon_k} \to +\infty, \text{ as } k \to +\infty.$$

Using (5.5.64) we see that

$$\left|\frac{z_k}{r_k}\right| \to 0, \text{ as } k \to +\infty, \tag{5.5.65}$$

and consequently, $\int\limits_{B_{\frac{r_k}{2}}(0)} |z|^{2\alpha} V_k e^{u_k} \le 8\pi(1+\alpha)$, provided that k is sufficiently large.

Since by (5.5.7) we can always verify (5.5.12) along a subsequence, we are in position to apply Corollary 5.4.24 to (a subsequence of) the following sequence:

$$u_{1,k}(z) = u_k(\frac{r_k}{2}z) + 2(\alpha+1)\log\frac{r_k}{2}, \; z \in B_1.$$

Thus we obtain

$$u_{1,k}(0) = \max_{|z|\le 1} u_{1,k} + O(1), \text{ as } k \to +\infty;$$

that is,

$$u_k(0) = \max_{|z|\le \frac{1}{2}r_k} u_k + O(1), \text{ as } k \to +\infty.$$

On the other hand, from (5.5.65) we see that $z_k \in B_{\frac{1}{2}r_k}(0)$ for large k, and we conclude

$$u_k(0) \ge u_k(z_k) - C,$$

for a suitable constant $C > 0$. But this is impossible, since the estimate above together with (5.5.63) leads to a contradiction of Theorem 5.5.39. □

Remark 5.5.43 Concerning the "inf+sup" estimate of (5.5.46), a first (weaker) version was established by Shafrir in [Sh] under the sole assumption:

$$0 < b_1 \le V \le b_2 \text{ in } \Omega.$$

In [Sh], Shafrir proves the existence of positive constants $C_1 = C_1(b/a)$ and $C_2 = C_2(a, b, \text{dist}\,(K, \partial\Omega))$ such that

$$\max_K u + C_1 \inf_\Omega u \le C_2.$$

His proof relies on the isoperimetrical inequality of Alexandroff (see [Ban]). Under the stronger assumption of (5.5.43), the sharper "inf+sup" inequality (5.5.46) was obtained by Brezis–Li–Shafrir [BLS] using a "moving-plane" technique that we previously adapted for the proof of Theorem 5.5.36.

We also note that in [Sh], Shafrir showed how to take advantage of the Liouville formula (2.2.3) to obtain (5.5.46) for the pure Liouville equation, where $\alpha = 0$ and $V \equiv 1$ in (5.5.42). We point out that when $\alpha > 0$, a similar use of the Liouville formula (as worked out in [BT1]) only enables one to derive (5.5.44). We give an indication of this fact for $\alpha \in (0, +\infty) \setminus \mathbb{N} \bigcup \{-\frac{1}{2} + \mathbb{N}\}$. In this case, a classification result in [BT1] asserts that all solutions for $-\Delta u = |z|^{2\alpha} e^u$ in B_1 take one of the following forms:

$$u(z) = \log\left(\frac{8|(1+\alpha)\psi(z) + z\psi'(z)|^2}{(1+|z|^{2(\alpha+1)}|\psi(z)|^2)^2}\right), \tag{5.5.66}$$

or

$$u(z) = \log\frac{8|(1+\alpha)\psi(z) - z\psi'(z)|^2}{(|z|^{2(\alpha+1)} + |\psi(z)|^2)^2}, \tag{5.5.67}$$

with ψ holomorphic in B_1 satisfying $\psi(0) \neq 0$ and $(1+\alpha)\psi(z) \pm z\psi'(z) \neq 0$ in B_1 where the $\pm$ sign is chosen according to whether (5.5.66) or (5.5.67) is considered. Thus, following [Sh], we define

$$v(z) = \log\left(\frac{8|(1+\alpha)\psi(z) \pm z\psi'(z)|^2}{(1+|\psi(z)|^2)^2}\right), z \in B_1,$$

where again the $\pm$ sign is chosen according to whether we use (5.5.66) or (5.5.67). Since v is superharmonic in B_1, we find

$$\log\frac{8(1+\alpha)^2|\psi(0)|^2}{(1+|\psi(0)|^2)^2} = v(0) \geq \min_{B_1} v = \min_{\partial B_1} v = \min_{B_1} u.$$

On the other hand, if u satisfies (5.5.66), then $u(0) = \log\left(8(1+\alpha)^2|\psi(0)|^2\right)$ and we conclude:

$$u(0) + \inf_{\partial B_1} u \leq \log\frac{64(1+\alpha)^4|\psi(0)|^4}{(1+|\psi(0)|^2)^2} \leq \log\left(64(1+\alpha)^4\right).$$

Whereas, if u satisfies (5.5.67), then $u(0) = \log\frac{8(1+\alpha)^2}{|\psi(0)|^2}$ and

$$u(0) + \inf_{\partial B_1} u \leq \log\frac{64(1+\alpha)^2}{(1+|\psi(0)|^2)^2} \leq \log\left(64(1+\alpha)^2\right).$$

In either case, (5.5.44) is established.

5.5.4 A Quantization property

The goal of this section is to establish the following result:

Theorem 5.5.44 *Let u_k satisfy (5.5.1), (5.5.2) and assume (5.5.6), (5.5.7) hold. Then (5.5.4) also holds.*

We start with some preliminaries. First notice that (5.5.1) and (5.5.2) imply that

$$\forall\, r \in (0,1), \quad \exists C_r > 0 : \int_{B_r} |z|^{2\alpha_k} V_k e^{u_k} \leq C_r, \tag{5.5.68}$$

$$\text{zero is the only blow-up point for } u_k \text{ in } B_1. \tag{5.5.69}$$

As already mentioned, Theorem 5.5.36 will play a crucial role in proving Theorem 5.5.44 as it implies the following result:

Proposition 5.5.45 *Under the assumption of Theorem 5.5.36, suppose further that for some $0 < \delta_k < r_k < 1/2$, we have*

$$\sup_{\{\frac{\delta_k}{2} \leq |z| \leq 2r_k\}} \{u_k + 2(\alpha + 1)\log|z|\} \leq C \text{ and } \frac{|z_k|}{\delta_k} \leq \gamma,$$

for suitable positive constants C and γ. Then

$$\int_{\{\delta_k \leq |z| \leq r_k\}} |z|^{2\alpha_k} V_k e^{u_k} \to 0, \text{ as } k \to +\infty. \tag{5.5.70}$$

Proof. As a consequence of Harnack's inequality as stated in Proposition 5.2.10, there exist $\beta \in (0,1)$ and $C > 0$ such that, for every $r \in (\delta_k, r_k)$, we have:

$$\sup_{|z|=r} u_k \leq \beta \inf_{|z|=r} u_k + 2(\alpha_k + 1)(\beta - 1)\log r + C. \tag{5.5.71}$$

Define

$$u_{k,r}(z) = u_k(rz) + 2(\alpha_k + 1)\log r.$$

We are going to apply Theorem 5.5.36 to $u_{k,r}$ with $\varepsilon_{0,r} = \varepsilon_0, z_{k,r} = \frac{1}{r} z_k$, and $V_{k,r}(z) = V_k(rz)$. Since

$$v_{k,r}(z) = u_{k,r}(|z_{k,r}|z) + 2(\alpha_k + 1)\log|z_{k,r}| = v_k(z)$$

we conclude that for a suitable constant C depending only on b_1, b_2 and A

(i) either $\max_{|z|=\leq\varepsilon_0} v_k \to -\infty$ and $\inf_{B_1} u_{k,r} \leq \max_{|z|\leq\varepsilon_0} v_k + 2(\alpha_k + 1)\log\frac{|z_k|}{r} + C$;

(ii) or $v_k(0) \to +\infty$ and $\inf_{B_1} u_{k,r} \leq -u_{k,r}(0) + C$.

In case (i), we find

$$\inf_{|z|=r} u_k \le \max_{|z|\le\varepsilon_0} v_k + 2(\alpha+1)\log|z_k| - 4(\alpha_k+1)\log r + C. \tag{5.5.72}$$

Hence, using (5.5.72) into (5.5.71), we derive the estimate

$$\begin{aligned}\int_{\{\delta_k<|z|<r_k\}} |z|^{2\alpha_k} V_k e^{u_k} &\le C e^{\beta \max_{|z|\le\varepsilon_0} v_k} |z_k|^{2(\alpha_k+1)\beta}\left(\frac{1}{\delta_k^{2(\alpha_k+1)\beta}} - \frac{1}{r_k^{2(\alpha_k+1)\beta}}\right)\\ &\le C\gamma^{\,2(\alpha_k+1)\beta} e^{\beta\max_{|z|\le\varepsilon_0} v_k} \to 0,\ \text{ as } k\to+\infty,\end{aligned}$$

and (5.5.70) is established in this case.

In case (ii), inequality (5.5.72) must be modified as

$$\inf_{|z|=r} u_k \le -u_k(0) - 4(\alpha_k+1)\log r + C, \tag{5.5.73}$$

and as above, (5.5.73) leads to the estimate

$$\begin{aligned}\int_{\{\delta_k<|z|<r_k\}} |z|^{2\alpha_k} V_k e^{u_k} &\le C e^{-\beta u_k(0)}\left(\frac{1}{\delta_k^{2(\alpha_k+1)\beta}} - \frac{1}{r_k^{2(\alpha_k+1)\beta}}\right)\\ &\le C\gamma^{\,2(\alpha_k+1)\beta} e^{-\beta v_k(0)} \to 0,\ \text{ as } k\to+\infty,\end{aligned}$$

and (5.5.70) is established in this case as well. □

Proposition 5.5.46 *Under the assumptions of Theorem 5.5.44, there exist* $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$ *and* $C > 0$ *such that, along a subsequence, we have*
(i) either

$$\sup_{|z|\le 2\varepsilon_0} \{u_k + 2(\alpha_k+1)\log|z|\} \le C; \tag{5.5.74}$$

(ii) or there exist sequences $\{z_{j,k}\} \subset B\setminus\{0\}$, $j = 1,\dots,m$, *satisfying*
1.

$$z_{j,k}\to 0\,,\ \ u_k(z_{j,k}) + 2(\alpha_k+1)\log|z_{j,k}| \to +\infty,\ \textit{as } k\to+\infty, \tag{5.5.75}$$

$\forall\ j = 1,\dots,m.$
2. Set $D_k = \{z : 0 < |z| \le 2\varepsilon_0|z_{1,k}|\} \cup \{z : |z| \ge \frac{1}{2\varepsilon_0}|z_{m,k}|\}$, *then*

$$\sup_{D_k}\{u_k + 2(\alpha_k+1)\log|z|\} \le C. \tag{5.5.76}$$

3. If $m \ge 2$, *then for every* $j = 1,\dots,m-1$*:*

$$\begin{gathered}\frac{|z_{j,k}|}{|z_{j+1,k}|} \to 0,\ \textit{as } k\to+\infty;\\ \sup_{\{\frac{1}{2\varepsilon_0}|z_{j,k}|\le|z|\le 2\varepsilon_0|z_{j+1,k}|\}} \{u_k(z) + 2(\alpha_k+1)\log|z|\} \le C.\end{gathered} \tag{5.5.77}$$

Proof. Accordingly to Proposition 5.4.26, there exist $\varepsilon_0 \in (0, \frac{1}{2})$ and a constant $C > 0$ such that, either alternative (5.5.74) holds or there exists a sequence $z_{1,k} \in B_1 \setminus \{0\}$ such that $z_{1,k} \to 0$, and (along a subsequence) $u_k(z_{1,k}) + 2(\alpha_k + 1) \log |z_{1,k}| \to +\infty$, as $k \to +\infty$. Moreover,

$$\sup_{\{|z| \leq 2\varepsilon_0 |z_{1,k}|\}} \{u_k(z) + 2(\alpha_k + 1) \log |z|\} \leq C.$$

For $\varepsilon \in (0, \frac{1}{2})$, repeat an analogous alternative in the set $\{|z| \geq \frac{1}{2\varepsilon}|z_{1,k}|\}$. By taking ε_0 smaller if necessary, we obtain either:

$$\sup_{\{|z| \geq \frac{1}{2\varepsilon_0}|z_{1,k}|\} \cap B_1} \{u_k(z) + 2(\alpha_k + 1) \log |z|\} \leq C,$$

which would yield to the desired statement with $m = 1$, or there exists a sequence $y_k \in B_1 \setminus \{0\}$ such that, as $k \to +\infty$,

$$\frac{|z_{1,k}|}{|y_k|} \to 0,$$

and

$$u_k(y_k) + 2(\alpha_k + 1) \log |y_k| \to +\infty. \tag{5.5.78}$$

Since zero is the only blow-up point for u_k in B_1, by necessity,

$$y_k \to 0. \tag{5.5.79}$$

In this second alternative, we are going to identify a second sequence $z_{2,k}$, by considering the extremal problem:

$$\sup_{\{\frac{1}{2\varepsilon_0}|z_{1,k}| \leq |z| \leq 2\varepsilon |y_k|\}} \{u_k(z) + 2(\alpha_k + 1) \log |z|\}, \tag{5.5.80}$$

for $\varepsilon \in (0, \frac{1}{2})$.

If the expression in (5.5.80) is uniformly bounded for some $\varepsilon \in (0, \frac{1}{2})$, then we simply take $z_{2,k} = y_k$, and adjust ε_0 accordingly in order to ensure the validity of (5.5.77) for $j = 1$. Otherwise, we obtain a new intermediate sequence of points with the same properties of (5.5.78) and (5.5.79), but infinitesimal with respect to y_k. Repeat the same analysis of (5.5.80) but with y_k replaced by such new sequence. As before, it may lead to a new sequence, in which case we continue in the same way. As in the proof of Proposition 5.4.26, each of such new sequences contributes by an amount of 8π to the value of β. So after a finite number of steps, we must obtain a sequence for which (5.5.80) is uniformly bounded for some $\varepsilon \in (0, \frac{1}{2})$. The sequence will define $z_{2,k}$, where we adjust $\varepsilon_0 \in (0, \frac{1}{2})$ in order to guarantee that (5.5.77) holds with $j = 1$. We iterate the argument above by replacing $z_{1,k}$ with the new sequence $z_{2,k}$. At this point, we are either able to check (5.5.75), (5.5.76), and (5.5.77) for $m = 2$, or obtain a third sequence for which we can verify (5.5.75) and (5.5.77) for $j = 1, 2$. We repeat the argument above for such new sequence, to either find that $m = 3$, or continue until we obtain (after a finite number of steps) the m-sequences that allow us to verify the desired properties. □

Alternative (i) in Proposition 5.5.46 is easy to handle as we have:

Proposition 5.5.47 *If the sequence u_k in Theorem 5.5.44 satisfies (5.5.74), then $\beta = 8\pi(1+\alpha)$.*

Proof. By Corollary 5.4.24, the validity of (5.5.74) implies that

$$u_k(0) = \max_{|z|\le 2\varepsilon_0} u_k + 0(1) \to +\infty, \text{ as } k \to +\infty. \tag{5.5.81}$$

So

$$\varepsilon_k = e^{-\frac{u_k(0)}{2(\alpha_k+1)}} \to 0, \text{ as } k \to +\infty,$$

and along a subsequence,

$$\xi_k(z) = u_k(\varepsilon_k z) + 2(\alpha_k + 1)\log\varepsilon_k \to \xi \text{ uniformly in } C^2_{\text{loc}}(\mathbb{R}^2),$$

with ξ as defined in (5.5.21)–(5.5.22).

Since

$$\int_{\mathbb{R}^2} |z|^{2\alpha} e^{\xi} = 8\pi(1+\alpha),$$

we find $R_k \to +\infty$ such that, along a subsequence,

$$\int_{\{|z|\le R_k\varepsilon_k\}} |z|^{2\alpha} V_k e^{u_k} \to 8\pi(1+\alpha), \text{ as } k \to +\infty.$$

For every $r \in (0, \varepsilon_0)$, we can use Proposition 5.2.10 to obtain the estimate

$$\sup_{\{|z|=r\}} u_k \le \beta \inf_{\{|z|=r\}} u_k + 2(\alpha_k + 1)(\beta - 1)\log r + C, \tag{5.5.82}$$

with $\beta \in (0, 1)$ and $C > 0$ independent of k and r.

Furthermore, we can apply Theorem 5.5.39 to $u_{k,r}(z) = u_k(rz) + 2(\alpha_k + 1)\log r$ to find

$$\inf_{|z|=r} u_k \le -u_k(0) - 4(\alpha_k + 1)\log r + C, \tag{5.5.83}$$

which, combined with (5.5.82), yields to the estimate

$$|z|^{2\alpha_k} V_k e^{u_k} \le \frac{Ce^{-\beta u_k(0)}}{r^{2(\alpha_k+1)\beta+1}}, \tag{5.5.84}$$

for $|z| = r$ and C independent of r and k.

Consequently, by (5.5.84) we derive

$$\int_{\{\varepsilon_k R_k \le |z| \le \varepsilon_0\}} |z|^{2\alpha_k} V_k e^{u_k} \le C e^{-\beta u_k(0)} \left(\frac{1}{(R_k \varepsilon_k)^{2(\alpha_k+1)\beta}} - \frac{1}{\varepsilon_0^{2(\alpha_k+1)\beta}} \right)$$
$$\le \frac{C}{R_k^{2(\alpha_k+1)\beta}} \to 0, \text{ as } k \to +\infty.$$

So

$$\int_{\{|z| \le \varepsilon_0\}} |z|^{2\alpha_k} V_k e^{u_k} = \int_{\{|z| \le \varepsilon_k R_k\}} |z|^{2\alpha_k} V_k e^{u_k} + o(1) = 8\pi(1+\alpha) + o(1),$$

and the desired conclusion follows by letting $k \to +\infty$. □

A last ingredient needed for the proof of Theorem 5.5.44 is the following result:

Lemma 5.5.48 *Let u_k satisfy (5.5.1), with $\alpha_k = 0$ and V_k satisfying (5.5.7). Suppose that (5.5.2) holds with $\beta < 16\pi$. For $r_0 \in (0,1)$, let $z_k \in \bar{B}_{r_0}$ such that $u_k(z_k) = \max_{\bar{B}_{r_0}} u_k(z_k)$. Then*

$$\max_{\bar{B}_{r_0}} \{u_k(z) + 2\log|z - z_k|\} < C, \tag{5.5.85}$$

and $\beta = 8\pi$.

Proof. In view of (5.5.2), we have that $z_k \to 0$ as $k \to +\infty$. So for k large, the function $\tilde{u}_k(z) = u_k(z_k + r_0 z) + 2\log r_0$ is well-defined in B_1 and satisfies

$$\begin{cases} -\Delta \tilde{u}_k = \tilde{V}_k e^{\tilde{u}_k}, \\ \tilde{u}_k(0) = \max_{B_1} \tilde{u}_k \to +\infty, \end{cases} \tag{5.5.86}$$

with $\tilde{V}_k(z) = V_k(z_k + r_0 z)$ satisfying (5.5.6) in B_1. Furthermore, $\tilde{V}_k e^{\tilde{u}_k} \rightharpoonup \beta\delta_{z=0}$, weakly in the sense of measure in B_1. Here β is the same value as that of u_k in (5.5.2). Since $\beta < 16\pi$, we claim that $\tilde{u}_k$ must satisfy alternative (i) of Proposition 5.5.46. Indeed, if by contradiction we assume that

$$\exists \tilde{z}_k \to 0 : \tilde{u}_k(\tilde{z}_k) + 2\log|\tilde{z}_k| \to +\infty \text{ as } k \to +\infty, \tag{5.5.87}$$

then

$$\liminf_{k\to+\infty} \int_{\{|z-\tilde{z}_k| < \delta|\tilde{z}_k|\}} \tilde{V}_k e^{\tilde{u}_k} \ge 8\pi, \quad \forall\, \delta > 0. \tag{5.5.88}$$

On the other hand, setting

$$\varepsilon_k = e^{-\frac{\tilde{u}_k(0)}{2}} \to 0, k \to +\infty,$$

and in view of (5.5.86), we can apply Lemma 5.4.21 to $\xi_k(z) = u_k(\varepsilon_k z) + 2\log\varepsilon_k$ and see that ξ_k satisfies (5.5.26). In terms of $\tilde{u}_k$, this implies the following:

$$\forall \varepsilon > 0, \exists R_\varepsilon > 0 : \int_{\{|z|\le R_\varepsilon \varepsilon_k\}} \tilde{V}_k e^{\tilde{u}_k} \ge 8\pi - \varepsilon, \tag{5.5.89}$$

for large $k \in \mathbb{N}$. Furthermore, from (5.5.86) and (5.5.87), we derive

$$\tilde{u}_k(0) + 2\log|\tilde{z}_k| \ge \tilde{u}_k(\tilde{z}_k) + 2\log|\tilde{z}_k| \to +\infty;$$

that is,

$$\frac{\varepsilon_k}{|\tilde{z}_k|} \to +\infty, \text{ as } k \to +\infty.$$

Therefore, the set $\{|z - \tilde{z}_k| < \delta|\tilde{z}_k|\} \cap B_{R\varepsilon_k}$ is empty for any $\delta \in (0,1)$ and $R > 1$, provided that k is sufficiently large.

But this is impossible, since (5.5.88) and (5.5.89) would imply that $\beta \ge 16\pi - \varepsilon$ for every $\varepsilon > 0$ and thus contradict our assumption of $\beta < 16\pi$.

Hence, we conclude that $\tilde{u}_k$ satisfies (5.5.74) (with $\alpha_k = 0$). Consequently, (5.5.85) holds, and we can apply Proposition 5.5.47 to $\tilde{u}_k$, to obtain $\beta = 8\pi$. □

We are finally ready to give:

Proof of Theorem 5.5.44. In view of Proposition 5.5.47, we only need to consider the case where alternative (ii) holds in Proposition 5.5.46. In this situation, we can apply Proposition 5.5.45 and derive

$$\int_{\{\varepsilon_0|z_{m,k}|\le|z|\le 1\}} |z|^{2\alpha_k} V_k e^{u_k} \to 0, \text{ as } k \to +\infty,$$

and for $m \ge 2$,

$$\int_{\{\frac{1}{\varepsilon_0}|z_{j,k}|\le|z|\le\varepsilon_0|z_{j+1,k}|\}} |z|^{2\alpha_k} V_k e^{u_k} \to 0, \text{ as } k \to +\infty; \forall\, j = 1,\dots,m-1.$$

Consequently,

$$\beta = \int_{\{|z|\le\varepsilon_0|z_{1,k}|\}} |z|^{2\alpha_k} V_k e^{u_k} + \sum_{j=1}^{m} \int_{\{\varepsilon_0|z_{j,k}|\le|z|\le\frac{1}{\varepsilon_0}|z_{j,k}|\}} |z|^{2\alpha_k} V_k e^{u_k} + o(1), \tag{5.5.90}$$

as $k \to +\infty$. Set

$$D_0 = \{z : \varepsilon_0 < |z| < \frac{1}{\varepsilon_0}\},$$

and define

$$v_{j,k}(z) = u_k(|z_{j,k}|z) + 2(\alpha_k + 1)\log|z_{j,k}|, \; z \in D_0, \tag{5.5.91}$$

for $j = 1, \dots, m$.

Then

$$-\Delta v_{j,k} = V_{j,k}(z)e^{v_{j,k}} \text{ in } D_0, \tag{5.5.92}$$

$$\int_{D_0} V_{j,k}(z)e^{v_{j,k}} \le C_0, \tag{5.5.93}$$

with $C_0 > 0$ a suitable constant. Also $V_{j,k}(z) = |z|^{2\alpha_k} V_k(|z_{j,k}|z)$ satisfies

$$0 < b_1 \le V_{j,k} \le b_2 \text{ and } |\nabla V_{j,k}| \le A \text{ in } D_0. \tag{5.5.94}$$

Moreover, passing to a subsequence if necessary, set

$$\beta_0 = \lim_{k\to+\infty} \int_{\{|z|\le\varepsilon_0\}} V_{1,k}e^{v_{1,k}}, \tag{5.5.95}$$

$$\beta_j = \lim_{k\to+\infty} \int_{D_0} V_{j,k}e^{v_{j,k}}. \tag{5.5.96}$$

So from (5.5.90), we find

$$\beta = \beta_0 + \sum_{j=1}^{m} \beta_j. \tag{5.5.97}$$

Concerning β_0, we have:

Claim:

$$\beta_0 = 8\pi(1+\alpha) \text{ or } \beta_0 = 0. \tag{5.5.98}$$

To establish (5.5.98), we can simply apply Theorem 5.5.36 (with $z_k = z_{1,k}$) to obtain that, either $\max\limits_{|z|\le\varepsilon_0} v_{1,k} \to -\infty$ and $\beta_0 = 0$ in this case, or $v_{1,k}(0) \to +\infty$ and $V_{1,k}(z)e^{v_{1,k}} \rightharpoonup \beta_0\delta_{z=0}$, weakly in the sense of measure in $B_{2\varepsilon_0}$. Since $\max\limits_{|z|\le2\varepsilon_0} \{v_{1,k} + 2(\alpha_k+1)\log|z|\} \le C$, we can use Proposition 5.5.47 for the sequence $v_{1,k}$ and conclude that $\beta_0 = 8\pi(1+\alpha)$ in this case.

Concerning the values β_j, note that (5.5.77) implies

$$\max_{D_0\setminus\{2\varepsilon_0\le|z|\le\frac{1}{2\varepsilon_0}\}} v_{j,k} \le C,$$

while (5.5.75) gives

$$v_{j,k}\left(\frac{z_{j,k}}{|z_{j,k}|}\right) \to +\infty, \text{ as } k \to +\infty.$$

Therefore, if S_j denotes the blow-up set of $v_{j,k}$ in D_0, then S_j is not empty and

$$S_j \subset \left\{2\varepsilon_0 \le |z| \le \frac{1}{2\varepsilon_0}\right\} \subset\subset D_0,\ j = 1,\dots,m. \tag{5.5.99}$$

Notice that property (5.5.99) remains valid for the blow-up set of any subsequence of $v_{j,k}$.

Hence, in view of (5.5.94) and (5.5.99), we need to focus only on the case for which blow-up occurs away from zero. In other words, after the usual translation and scaling, it remains to analyze the situation where u_k satisfies:

$$\begin{cases} -\Delta u_k = V_k e^{u_k} & \text{in } B_1, \\ V_k e^{u_k} \to \beta \delta_{z=0}, & \text{weakly in the sense of measure.} \end{cases} \tag{5.5.100}$$

Claim: If u_k satisfies (5.5.100), with V_k satisfying (5.5.6), then

$$\beta \in 8\pi \mathbb{N}. \tag{5.5.101}$$

This result was proved by Li-Shafrir in [LS]. To establish (5.5.101), we can collect the information obtained thus far for $\alpha_k = 0$, which together with Lemma 5.5.48 allows us to assert the following:
(I) either $\beta = 8\pi$

(II) or $\beta \geq 16\pi$ and there exist $\varepsilon_0 > 0$ and sequences $v_{j,k}$ $j = 1, \ldots, m$ such that in $D_0 = \{\varepsilon_0 < |z| < \frac{1}{\varepsilon_0}\}$ they satisfy (5.5.92), (5.5.93), (5.5.94), (5.5.99), and $\beta = \beta_0 + \sum_{j=1}^{m} \beta_j$ with $\beta_0 = 0$ or 8π and β_j defined by (5.5.96).

Notice that if (II) holds, then we can use Proposition 5.4.32 (c) to apply the alternative above (at the blow-up point zero) to the following sequence:

$$u_k^j(z) = v_{j,k}(z_0 + r_0 z) + 2 \log r_0, \ z \in B_1 \tag{5.5.102}$$

(possibly along a subsequence), for every $z_0 \in S_j$, $j = 1, \ldots, m$ and $r_0 > 0$ sufficiently small. In fact, each time (II) holds, we can construct new sequences to which we can apply again the above alternative. On the other hand, each time that alternative (II) holds for any of such sequences it contributes by an amount of at least 16π to the value β in (5.5.100). Therefore, after a finite number of steps, we can no longer suppose the validity of (II), and we end up with finitely many sequences for which (I) holds. This proves the claim.

At this point we can complete the proof of (5.5.4), just by applying the claim to (a subsequence of) u_k^j defined in (5.5.102) with $v_{j,k}$ in (5.5.91) and any blow-up point $z_0 \in S_j$. Consequently, we find that $\beta_j \in 8\pi \mathbb{N}$, and we derive (5.5.4) by taking into account Proposition 5.5.71 together with (5.5.97) and (5.5.98). □

5.5.5 Examples

To illustrate the content (and sharpness) of Theorem 5.5.44, in this section we present some instructive examples.

First if we take $f(z) = \frac{z^{\alpha+1}\phi(z)}{\lambda \psi(z)}$ in Liouville's formula (2.2.3), with ϕ and ψ holomorphic functions that are non-vanishing at the origin and with $\lambda \in \mathbb{R}$, then

$$u_\lambda(z) = \log\left(\frac{8\lambda^2|(\alpha+1)\phi(z)\psi(z) + z(\phi'(z)\psi(z) - \phi(z)\psi'(z))|^2}{\left(\lambda^2|\psi(z)|^2 + |\phi(z)|^2|z|^{2(\alpha+1)}\right)^2}\right) \quad (5.5.103)$$

defines a solution for

$$-\Delta u = |z|^{2\alpha}e^u, \quad (5.5.104)$$

in a domain D where $(\alpha+1)\phi(z)\psi(z) + z\left(\phi'(z)\psi(z) - \phi(z)\psi'(z)\right)$ never vanishes.

By suitable choices of ψ, ϕ and λ, we are able to construct solution sequences u_k for (5.5.104) satisfying

$$|z|^{2\alpha}e^{u_k} \rightharpoonup 8\pi m\delta_{z=0}, \text{ weakly in the sense of measure in } B_1, \quad (5.5.105)$$

for any given $m \in \mathbb{N}$; or

$$|z|^{2\alpha}e^{u_k} \rightharpoonup (8\pi(1+\alpha) + 8\pi m)\delta_{z=0}, \text{ weakly in the sense of measure in } B_1, \quad (5.5.106)$$

for any given $m \in \mathbb{N} \cup \{0\}$.

Our method is inspired by the construction given by X.X. Chen in [Chn] to obtain (5.5.105) in case $\alpha = 0$. We start with (5.5.105). In (5.5.103), take

$$\phi(z) = 1, \quad \psi(z) = (z^m - 1)e^{g(z)}, \quad (5.5.107)$$

with g holomorphic in B_1 and $g(0) = 0$ such that

$$(\alpha+1)(z^m - 1) - mz^m - z(z^m - 1)g'(z) = -(\alpha+1)e^{z^m \log\left(\frac{m}{\alpha+1}\right)} \quad (5.5.108)$$

Namely, $g(z)$ in the holomorphic function over $\mathbb{C}$ defined by the conditions:

$$\begin{cases} g'(z) = \frac{(\alpha+1)e^{z^m \log\left(\frac{m}{\alpha+1}\right)} - mz^m + (\alpha+1)(z^m-1)}{z(z^m-1)}, \\ g(0) = 0. \end{cases} \quad (5.5.109)$$

Notice that the right-hand side of (5.5.109) is also well-defined at $z = 0$ and at the m-complex roots of unity:

$$z_j = e^{\frac{2\pi j}{m}i}, \quad j = 0, 1 \ldots, m-1. \quad (5.5.110)$$

Consequently, for every $\lambda \in \mathbb{R}$,

$$v_\lambda(z) = \log\left(\frac{8(\alpha+1)^2\lambda^2|e^{g(z)}|^2|e^{z^m \log\left(\frac{m}{\alpha+1}\right)}|^2}{\left(|z|^{2(\alpha+1)} + \lambda^2|z^m - 1|^2|e^{g(z)}|^2\right)^2}\right) \quad (5.5.111)$$

defines a solution for (5.5.104) in the whole complex plane. Our next task is to determine a sequence $\lambda_k \to +\infty$ such that

$$\int_{\{|z|<k\}} |z|^{2\alpha}e^{v_{\lambda_k}} \to 8\pi m, \text{ as } k \to +\infty. \quad (5.5.112)$$

For every $\varepsilon \in (0,1)$, choose $\delta_\varepsilon > 0$ sufficiently small so that the balls $B_{\delta_\varepsilon}(z_j)$ are mutually disjoint for every $j = 0, 1, \ldots, m-1$, and the following hold:

$$\begin{aligned}
&(1-\varepsilon)|e^{g(z_j)}|^2 < |e^{g(z)}|^2 < (1+\varepsilon)|e^{g(z_j)}|^2,\\
&(1-\varepsilon)^2 < |z|^{2(\alpha+1)} < (1+\varepsilon)^2,\\
&(1-\varepsilon)^3 m^2 < \left|(\alpha+1)e^{z^m \log\left(\frac{m}{\alpha+1}\right)}\right|^2 < (1+\varepsilon)^3 m^2,\\
&(1-\varepsilon)m^2 < \left|\frac{z^m-1}{z-z_j}\right|^2 < (1+\varepsilon)m^2,
\end{aligned}$$

for every $z \in B_{\delta_\varepsilon}(z_j)$, and $\forall j = 0, 1, \ldots, m-1$. Set $\sigma_j = m\left|e^{g(z_j)}\right|$ and $r_{j,\varepsilon} = \delta_\varepsilon \sigma_j$. By virtue of those estimates, we find

$$\begin{aligned}
8\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^4 \int\limits_{\{|z|<r_{j,\varepsilon}\lambda\}} \frac{1}{(1+|z|^2)^2} &\le \int\limits_{B_{\delta_\varepsilon}(z_j)} |z|^{2\alpha} e^{v_\lambda(z)}\\
&\le 8\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^4 \int\limits_{\{|z|<r_{j,\varepsilon}\lambda\}} \frac{1}{(1+|z|^2)^2}.
\end{aligned} \tag{5.5.113}$$

Applying (5.5.113) and recalling that

$$\int\limits_{\{|z|<r_{j,\varepsilon}\lambda\}} \frac{1}{(1+|z|^2)^2} \to \pi, \text{ as } \lambda \to +\infty,$$

we may conclude that $\forall \varepsilon > 0$, there exists $\delta_\varepsilon > 0$ and $\lambda_\varepsilon > 1$ such that $\forall \lambda > \lambda_\varepsilon$,

$$\int\limits_{B_{\delta_\varepsilon}(z_j)} |z|^{2\alpha} e^{v_\lambda(z)} = 8\pi + O(\varepsilon)\,,\ \forall j = 0, 1, \ldots, m-1.$$

On the other hand, in $\Omega_{R,\delta} = B_R \setminus \overline{\bigcup_{j=0}^{m-1} B_\delta(z_j)}$, $\quad R > 1$, we have:

$$\int\limits_{\Omega_{R,\delta}} |z|^{2\alpha} e^{v_\lambda} \le \frac{8\pi(\alpha+1)^2 R^{2(\alpha+1)} e^{R^m \log\left(\frac{m}{\alpha+1}\right)}}{\lambda^2 \delta^{4m} \min\limits_{|z|\le R} |e^{g(z)}|^2}. \tag{5.5.114}$$

Hence, by choosing $\varepsilon = \frac{1}{k}$ and $R = k$, we find $\delta_k \to 0$ and $\lambda_k \to +\infty$ such that, as $k \to +\infty$, we have

$$\int\limits_{\bigcup_{j=1}^{m-1} B_{\delta_k}(z_j)} |z|^{2\alpha} e^{v_{\delta_k}} = 8\pi m + o(1),$$

$$\int_{B_k \setminus \bigcup_{j=1}^{m-1} B_{\delta_k}(z_j)} |z|^{2\alpha} e^{v_\lambda} = o(1).$$

In particular notice that, from (5.5.114), by necessity:

$$\frac{\lambda_k}{k^{\alpha+1}} \to +\infty. \tag{5.5.115}$$

In B_1 define,

$$u_k(z) = v_{\lambda_k}(kz) + 2(\alpha+1)\log k;$$

that is,

$$u_k(z) = \log \frac{8(\alpha+1)^2 k^{2(\alpha+1)} \lambda_k^2 |e^{g(kz)}|^2 \left| e^{(kz)^m \log\left(\frac{m}{\alpha+1}\right)} \right|^2}{(k^{2(\alpha+1)}|z|^{2(\alpha+1)} + \lambda_k^2 |(kz)^m - 1|^2 |e^{g(kz)}|^2)^2}.$$

Hence, u_k satisfies (5.5.106) in B_1 and based on our choice of λ_k:

$$u_k(z_j/k) = \log\left(8k^{2(\alpha+1)}\lambda_k^2 \sigma_j^2\right) \to +\infty, \forall j = 0, 1, \ldots, m-1,$$

$$\int_{B_1} |z|^{2\alpha} e^{u_k} = \int_{B_k} |z|^{2\alpha} e^{v_{\lambda_k}} = 8\pi m + o(1),$$

$$\sup_{r \le |z| \le 1} |z|^{2\alpha} e^{u_k} \to 0, \text{ for every } r \in (0, 1),$$

as $k \to +\infty$. Thus, u_k verifies (5.5.105).

Remark 5.5.49 Observe that, although zero is a blow-up point for u_k,

$$u_k(0) = \log 8(\alpha+1)^2 \frac{k^{2(\alpha+1)}}{\lambda_k^2} \to -\infty, \text{ as } k \to +\infty,$$

as follows from (5.5.115).

In order to construct a sequence satisfying (5.5.106), we proceed in an analogous way. For $m = 0$, just take $\lambda_k \to +\infty$, and let

$$u_k(z) = \log \frac{8(\alpha+1)^2 \lambda_k^2}{(1 + \lambda_k^2 |z|^{2(\alpha+1)})^2}.$$

This function satisfies (5.5.104) in B_1, in addition to the following properties:
(i) $u_k(0) = \log 8(\alpha+1)^2 \lambda_k^2 \to +\infty$, as $k \to +\infty$;

(ii) $\int_{\{|z|\le 1\}} |z|^{2\alpha} e^{u_k} = \int_{\{|z|\le 1\}} \frac{8(\alpha+1)^2 |z|^{2\alpha} \lambda_k^2}{(1+\lambda_k^2 |z|^{2(\alpha+1)})^2} = 8(1+\alpha)^2 \int_{\{|z|\le \lambda_k^{1/1+\alpha}\}} \frac{|z|^{2\alpha}}{(1+|z|^{2(\alpha+1)})^2}$;

(iii) $\sup_{\{r \le |z| \le 1\}} |z|^{2\alpha} e^{u_k} = O\left(\frac{1}{\lambda_k}\right)$, for every $r \in (0, 1)$.

Since

$$\int_{\{|z|\le\lambda_k^{1/1+\alpha}\}} \frac{|z|^{2\alpha}}{(1+|z|^{2(\alpha+1)})^2} \to \int_{\mathbb{R}^2} \frac{|z|^{2\alpha}}{(1+|z|^{2(\alpha+1)})^2} = \frac{\pi}{\alpha+1}, \text{ as } k \to +\infty,$$

and in view of the properties above, we promptly verify that u_k satisfies (5.5.106) with $m = 0$.

For $m \in \mathbb{N}$, in (5.5.103) take

$$\psi(z) = \frac{1}{\lambda} \text{ and } \phi(z) = \lambda(z^m - 1)e^{g(z)},$$

with $g(z)$ the holomorphic function defined by the conditions:

$$\begin{cases} g'(z) = -\frac{(\alpha+1)e^{z^m\left(\log\frac{m}{\alpha+1}+i\pi\right)}+mz^m+(\alpha+1)(z^m-1)}{z(z^m-1)}, \\ g(0) = 0. \end{cases}$$

Hence,

$$v_\lambda(z) = \log \frac{8(\alpha+1)^2\lambda^2|e^{g(z)}|^2|e^{z^m\left(\log\frac{m}{\alpha+1}+i\pi\right)}|^2}{(1+\lambda^2|z|^{2(\alpha+1)}|z^m-1|^2|e^{g(z)}|^2)^2}$$

satisfies (5.5.104) in the complex plane.

Similarly to the case (5.5.111), we can establish that $\forall\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ small and $\lambda_\varepsilon > 1$:

$$\int_{\bigcup_{j=0}^{m-1} B_{\delta_\varepsilon}(z_j)} |z|^{2\alpha}e^{v_\lambda} = 8\pi m + O(\varepsilon), \tag{5.5.116}$$

for every $\lambda \ge \lambda_\varepsilon$, with z_j defined in (5.5.110). Moreover, for $\delta > 0$ small, set $D_\delta = \bigcup_{j=0}^{m-1} B_\delta(z_j) \cup B_\delta(0)$. We then have

$$\int_{B_R\setminus D_\delta} |z|^{2\alpha}e^{v_\lambda} \le \frac{8\pi(1+\alpha)^2R^{2(\alpha+1)}e^{R^m\log\frac{m}{\alpha+1}}}{\lambda^2\delta^{4(m+1)}\min\limits_{|z|\le R}|e^{g(z)}|^2}. \tag{5.5.117}$$

On the other hand, around the origin, we see that for any given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every $z \in B_{\delta_\varepsilon}(0)$,

$$1-\varepsilon < \left|e^{g(z)+z^m\left(\log\frac{m}{\alpha+1}+i\pi\right)}\right|^2 < 1+\varepsilon,$$

$$1-\varepsilon < \left|(z^m-1)e^{g(z)}\right|^2 < 1+\varepsilon.$$

Consequently, in $B_{\delta_\varepsilon}(0)$ the following estimate holds:

$$\frac{8(\alpha+1)^2(1-\varepsilon)\lambda^2|z|^{2\alpha}}{(1+(1+\varepsilon)\lambda^2|z|^{2(\alpha+1)})^2} \le |z|^{2\alpha}e^{v_\lambda} \le \frac{8(1+\alpha)^2(1+\varepsilon)\lambda^2|z|^{2\alpha}}{(1+(1-\varepsilon)\lambda^2|z|^{2(\alpha+1)})^2}.$$

Letting $r_\varepsilon^\pm = \delta_\varepsilon(1\pm\varepsilon)^{\frac{1}{2(1+\alpha)}}$, we find

$$8(\alpha+1)^2 \int\limits_{B_{\delta_\varepsilon}(0)} \frac{\lambda^2|z|^{2\alpha}}{(1+(1\pm\varepsilon)\lambda^2|z|^{2(\alpha+1)})^2}$$

$$= 8(\alpha+1)^2 \int\limits_{\{|z|\le r_\varepsilon^\pm\lambda^{1/1+\alpha}\}} \frac{|z|^{2\alpha}}{(1+|z|^{2(\alpha+1)})^2} \to 8\pi(1+\alpha), \text{ as } \lambda\to+\infty.$$

Thus, for $\delta_\varepsilon > 0$ sufficiently small and $\lambda_\varepsilon > 1$ sufficiently large, we can also ensure that

$$\int\limits_{B_{\delta_\varepsilon}(0)} |z|^{2\alpha}e^{v_\lambda} = 8\pi(1+\alpha) + O(\varepsilon),\ \forall\lambda\ge\lambda_\varepsilon. \tag{5.5.118}$$

At this point, we can combine (5.5.116), (5.5.117), and (5.5.118) to find a sequence $\lambda_k \to +\infty$, such that

$$\int\limits_{\{|z|\le k\}} |z|^{2\alpha}e^{v_{\lambda_k}} = 8\pi(1+\alpha) + 8\pi m + O(\frac{1}{k}),\ \text{as } k\to+\infty.$$

Thus, exactly as above, we see that

$$u_k(z) = v_{\lambda_k}(kz) + 2(\alpha+1)\log k$$

verifies (5.5.106).

5.6 The effect of boundary conditions

5.6.1 Preliminaries

In this Section we are going to discuss the (strong) effect that the boundary condition (5.4.36) implies on the blow-up behavior for a solution sequence of (5.5.1), (5.5.2). First of all, we notice that a comparison between Theorem 5.4.28 and Proposition 5.5.47 suggests a connection between the conditions

$$\max_{\partial B_1} u_k - \min_{\partial B_1} u_k \quad \le C \tag{5.6.1}$$

and

$$\sup_{\{|z|<r\}} \{u_k(z) + 2(\alpha_k+1)\log|z|\} \le C \tag{5.6.2}$$

for some $r \in (0,1)$.

In fact, on the basis of Proposition 5.5.46, we can establish the following:

Proposition 5.6.50 *Let u_k satisfy (5.5.1), (5.5.2) with V_k as in (5.5.6) and $\alpha_k \to \alpha \in (0, +\infty) \setminus \mathbb{N}$. Then (5.6.1) implies (5.6.2), for any $r \in (0, 1)$.*

Proof. Suppose that u_k satisfies (5.6.1). Then by Theorem 5.4.28, (5.5.2) must hold with $\beta = 8\pi(1+\alpha)$. On the other hand, if alternative (ii) of Proposition 5.5.46 holds, then by the proof of Theorem 5.5.44, we would find that either $\beta \in 8\pi\mathbb{N}$ (corresponding to the case $\beta_0 = 0$) or $\beta \in 8\pi(1+\alpha) + 8\pi\mathbb{N}$ (corresponding to the case $\beta_0 = 8\pi(1+\alpha)$). In either case, we could not satisfy the condition

$$\beta = 8\pi(1+\alpha), \text{ with } \alpha \in (0, +\infty) \setminus \mathbb{N}.$$

Therefore alternative (i) of Proposition 5.5.46 must hold for u_k.

Taking into account that zero is the only blow-up point for u_k in B_1, we can actually conclude that (5.6.2) holds for every $r \in (0, 1)$. □

Such a result is *false* if we take $\alpha = N \in \mathbb{N} \cup \{0\}$. As an easy check, consider the sequence

$$u_k(z) = \log \frac{\lambda_k^2}{\left(1 + \gamma_N \lambda_k^2 \left|z^{N+1} - z_k^{N+1}\right|^2\right)^2}, \tag{5.6.3}$$

where $\gamma_N = \frac{1}{8(N+1)^2}$ and where $\{z_k\} \subset B_1$, $\lambda_k > 0$ satisfy

$$z_k \to 0, \quad \lambda_k |z_k|^{N+1} \to +\infty, \text{ as } k \to +\infty.$$

Indeed, by virtue of (2.2.13) and (2.2.15), we see that u_k satisfies (5.5.1) with $V_k = 1$ and $\alpha_k = N$, as well as, (5.5.2) and the boundary condition (5.6.1). In contrast, (5.6.2) fails for every $r \in (0, 1)$. We thus have:

$$z_k \to 0, \quad u_k(z_k) + 2(N+1)\log|z_k| = 2\log \lambda_k |z_k|^{N+1} \to +\infty, k \to +\infty.$$

It is also interesting to note that u_k blows up along the $N+1$ sequences: $z_{j,k} = z_k e^{\frac{2\pi j}{N+1} i}$, $j = 0, 1, 2 \ldots, N$. More precisely, setting

$$v_k(z) = u_k(|z_k| z) + 2(N+1)\log|z_k|,$$

and

$$p_0 = \lim_{k \to +\infty} \frac{z_k}{|z_k|},$$

(take a subsequence if necessary), we obtain

$$|z|^{2N} e^{v_k} \rightharpoonup 8\pi \sum_{j=0}^{N} \delta_{p_j}, \text{ with } p_j = e^{\frac{2\pi j}{N+1} i} p_0,$$

weakly in the sense of measure. We shall see that such a "multi-peak" profile cannot occur when $\alpha > 0$ is *not* an integer (see Corollary 5.6.56).

In fact, to strenghten even further the connection between (5.6.1) and (5.6.2) in Section 5.6.3 below, we shall prove that the strong "inf + sup" estimate (5.5.61) holds for functions subject to the boundary condition (5.6.1).

See Theorem 5.6.59 below and compare it with Corollary 5.5.42.

5.6.2 Pointwise estimates of the blow-up profile

The goal of this section is to provide pointwise estimates for solution sequences $u_k \in C^2(B_1) \cap C^0(\bar{B}_1)$ satisfying

$$\begin{cases} -\Delta u_k = |z|^{2\alpha_k} V_k e^{u_k} & \text{in } B_1, \\ \max\limits_{\partial B_1} u_k - \min\limits_{\partial B_1} u_k \le c_0, & \\ |z|^{2\alpha_k} V_k e^{u_k} \to \beta \delta_{z=0}, & \text{weakly in the sense of measure in } \bar{B}_1 \end{cases} \tag{5.6.4}$$

with $c_0 > 0$ a suitable constant.

Following the approach of Bartolucci–Chen–Lin–Tarantello [BCLT], we have:

Theorem 5.6.51 *Let u_k satisfies (5.6.4) and assume that (5.5.6) and (5.5.7) hold. If*

$$u_k(0) = \max_{\bar{B}_1} u_k + O(1), \tag{5.6.5}$$

then along a subsequence, we have

$$\left| u_k(z) - \log \frac{e^{u_k(0)}}{\left(1 + \gamma_\alpha V_k(0) e^{u_k(0)} |z|^{2(\alpha_k+1)}\right)^2} \right| \le C, \tag{5.6.6}$$

$\forall z \in \bar{B}_1$, *with $\gamma_\alpha = \frac{1}{8(\alpha+1)^2}$ and a suitable constant $C > 0$.*

Observe that, when $\alpha \in (0, +\infty) \setminus \mathbb{N}$, then (5.6.5) holds automatically (see Corollary 5.6.56).

On the other hand, when $\alpha \in \mathbb{N}$, example (5.6.3) shows that assumption (5.6.5) is necessary for the validity of (5.6.6).

In order to establish Theorem 5.6.51, we start with some preliminary observations. First of all, by passing to a subsequence if necessary, we are going to suppose that (5.5.12) and (5.5.13) hold.

Set

$$\varepsilon_k = e^{-\frac{u_k(0)}{2(\alpha_k+1)}} \tag{5.6.7}$$

and note that $\varepsilon_k \to 0$, as $k \to +\infty$, since u_k blows up in B_1 and (5.6.5) holds. Define

$$\xi_k(z) = u_k(\varepsilon_k z) + 2(\alpha_k + 1) \log \varepsilon_k \tag{5.6.8}$$

in $\bar{B}_{1/\varepsilon_k}$. Then $\xi_k(0) = 0$ and

$$\max_{\bar{B}_{1/\varepsilon_k}} \xi_k = \max_{\bar{B}_1} u_k - u_k(0) = O(1). \tag{5.6.9}$$

So by Lemma 4.1.2, we have that

$$\xi_k \text{ is uniformly bounded in } L^\infty_{\text{loc}}(\mathbb{R}^2), \tag{5.6.10}$$

and that along a subsequence,

$$\xi_k \to \xi \text{ uniformly in } C^2_{\text{loc}}(\mathbb{R}^2), \tag{5.6.11}$$

where

$$\xi(z) = \log \frac{\lambda_0}{(1+\gamma_\alpha \lambda_0 |z^{\alpha+1} - y_0|^2)^2}, \tag{5.6.12}$$

with $\gamma_\alpha = \frac{1}{8(1+\alpha)^2}$ and with $\lambda_0 \geq 1$, $y_0 \in \mathbb{C}$ satisfying

$$\begin{cases} \lambda_0 = \left(1+\gamma_\alpha \lambda_0 |y_0|^2\right)^2, & \text{for } \alpha \in \mathbb{N} \cup \{0\} \\ \lambda_0 = 1,\, y_0 = 0, & \text{for } \alpha \in (0,+\infty) \setminus \mathbb{N} \end{cases} \tag{5.6.13}$$

(see Remark 5.4.22). Our goal is to take advantage of the boundary condition in (5.6.5) in order to complete (5.6.11) with the global estimate

$$|\xi_k(z) - \xi(z)| \leq C \text{ in } \bar{B}_{1/\varepsilon_k},$$

for a suitable $C > 0$. To this purpose, notice that the function

$$\varphi_k(z) = u_k(z) - \min_{\partial B_1} u_k$$

satisfies

$$\begin{cases} -\Delta \varphi_k = |z|^{2\alpha_k} V_k e^{u_k} \text{ in } B_1, \\ 0 \leq \varphi_k \leq C \text{ on } \partial B_1. \end{cases}$$

So we can use Green's representation formula for φ_k to find

$$\varphi_k(z) = \frac{1}{2\pi} \int_{B_1} \left(\log \frac{1}{|z-y|}\right) |y|^{2\alpha_k} V_k(y) e^{u_k(y)} + \phi_k(y), \tag{5.6.14}$$

with ϕ_k uniformly bounded in $C^0(\bar{B}_1) \cap C^2_{\text{loc}}(B_1)$. Consequently,

$$\xi_k(z) = u_k(\varepsilon_k z) - u_k(0) = \frac{1}{2\pi} \int_{B_1} \left(\log \frac{|y|}{|\varepsilon_k z - y|}\right) |y|^{2\alpha_k} V_k(y) e^{u_k(y)} \\ + \phi_k(\varepsilon_k z) - \phi_k(0)$$

Hence, setting

$$\psi_k(z) = \phi_k(\varepsilon_k z) - \phi_k(0) \tag{5.6.15}$$

and after a change of variable, we derive

$$\xi_k(z) = \frac{1}{2\pi} \int_{\{|z| < \frac{1}{\varepsilon_k}\}} \left(\log \frac{|y|}{|y-z|}\right) |y|^{2\alpha_k} V_k(\varepsilon_k y) e^{\xi_k(y)} + \psi_k(z), \tag{5.6.16}$$

$\forall z \in B_{1/\varepsilon_k}$.

Lemma 5.6.52 *For every $\varepsilon > 0, \exists\, R_\varepsilon > 1, k_\varepsilon \in \mathbb{N}$ and $C_\varepsilon > 0$ such that, along a subsequence (denoted the same way), we have:*

$$\xi_k(z) \le (4(\alpha+1) - \varepsilon) \log \frac{1}{|z|} + C_\varepsilon, \tag{5.6.17}$$

$\forall |z| \ge 2R_\varepsilon$ *and* $\forall k \ge k_\varepsilon$.

Proof. We consider the subsequence for which (5.6.11) holds together with (5.5.12) and (5.5.13). By Theorem 5.4.28, we know that $\beta = 8\pi(1+\alpha)$ in (5.6.4).

Set

$$M_k = \int_{B_1} |z|^{2\alpha_k} V_k(z) e^{u_k(z)} = \int_{B_{1/\varepsilon_k}} |z|^{2\alpha_k} V_k(\varepsilon_k z) e^{\xi_k}, \tag{5.6.18}$$

so that

$$M_k \to 8\pi(1+\alpha), \text{ as } k \to +\infty. \tag{5.6.19}$$

Also recall that $\int_{\mathbb{R}^2} |z|^{2\alpha} e^{\xi} = 8\pi(1+\alpha)$, for ξ the limiting function in (5.6.11). Consequently, for a given $\varepsilon > 0$, we find $k_\varepsilon \in \mathbb{N}$ and $R_\varepsilon > 1$ such that the following hold:

$$\int_{B_{R_\varepsilon}} |z|^{2\alpha} e^{\xi} \ge 8\pi(1+\alpha) - \frac{2\pi\varepsilon}{5(\alpha+2)}; \tag{5.6.20}$$

and for $k \ge k_\varepsilon$,

$$|M_k - 8\pi(1+\alpha)| < \frac{2\pi\varepsilon}{5(\alpha+2)}, \tag{5.6.21}$$

$$\int_{B_{R_\varepsilon}} |z|^{2\alpha_k} V_k(\varepsilon_k z) e^{\xi_k} \ge \int_{B_{R_\varepsilon}} |z|^{2\alpha} e^{\xi} - \frac{2\pi\varepsilon}{5(\alpha+2)}$$
$$\ge 2\pi\left(4(1+\alpha) - \frac{2\varepsilon}{5(\alpha+2)}\right). \tag{5.6.22}$$

In particular,

$$\int_{B_{1/\varepsilon_k} \setminus B_{R_\varepsilon}} |z|^{2\alpha_k} V_k(\varepsilon_k z) e^{\xi_k} < \frac{6\pi\varepsilon}{5(\alpha+2)}, \forall k \ge k_\varepsilon. \tag{5.6.23}$$

To obtain (5.6.17), we use expressions (5.6.16) and (5.6.22) so that for $z : |z| > 2R_\varepsilon$ and $\forall k \ge k_\varepsilon$, we can estimate

$$\frac{1}{2\pi}\int_{B_{R_\varepsilon}} \left(\log \frac{|y|}{|z-y|}\right) |y|^{2\alpha_k} V_k(\varepsilon_k y) e^{\xi_k(y)} \le \frac{1}{2\pi}\left(\log \frac{2R_\varepsilon}{|z|}\right) \int_{B_{R_\varepsilon}} |y|^{2\alpha_k} V_k(\varepsilon_k y) e^{\xi_k(y)}$$
$$\le \left(4(1+\alpha) - \frac{2\varepsilon}{5(\alpha+2)}\right) \log \frac{2R_\varepsilon}{|z|};$$

and we can derive

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\{|y|<\frac{|z|}{2}\}}\left(\log\frac{|y|}{|z-y|}\right)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k}\\
&\quad\leq \frac{1}{2\pi}\int_{B_{R_\varepsilon}}\left(\log\frac{|y|}{|z-y|}\right)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k}\\
&\qquad+\frac{1}{2\pi}\int_{\{R_\varepsilon<|y|<\frac{|z|}{2}\}}\left(\log\frac{|y|}{|z-y|}\right)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)} \qquad (5.6.24)\\
&\quad\leq \frac{1}{2\pi}\int_{B_{R_\varepsilon}}\left(\log\frac{|y|}{|z-y|}\right)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k}\leq\left(4(1+\alpha)-\frac{2\varepsilon}{5(\alpha+2)}\right)\log\frac{2R_\varepsilon}{|z|}\\
&\quad\leq\left(4(1+\alpha)-\frac{2\varepsilon}{5(\alpha+2)}\right)\log\frac{1}{|z|}+C_{1,\varepsilon},
\end{aligned}
$$

for a suitable constant $C_{1,\varepsilon}$.

Furthermore, we can use (5.6.9) and (5.6.23) to estimate:

$$
\begin{aligned}
&\int_{\{|z-y|<\frac{|z|}{2}\}}\left(\log\frac{|y|}{|z-y|}\right)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)}\\
&\quad=\int_{\{|z-y|<\frac{|z|}{2}\}}(\log|y|)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)}\\
&\qquad+\int_{\{|z-y|<\frac{1}{|z|^{\alpha+1}}\}}\left(\log\frac{1}{|z-y|}\right)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)}\\
&\qquad+\int_{\{\frac{1}{|z|^{\alpha+1}}\leq|y-z|<\frac{|z|}{2}\}}\left(\log\frac{1}{|z-y|}\right)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)}\\
&\quad\leq C\left(|z|+\frac{1}{|z|^{\alpha+1}}\right)^{2\alpha_k}\int_{\{|z-y|<\frac{1}{|z|^{\alpha+1}}\}}\log\frac{1}{|z-y|}dy\\
&\qquad+\Big((\alpha+2)\log|z|+O(1)\Big)\int_{\{|y|\geq\frac{|z|}{2}\}}|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)}\\
&\quad\leq 2\pi\left(C\left(\frac{1}{R_\varepsilon}\right)^{2(1+\alpha-\alpha_k)}+\frac{3}{5}\varepsilon\right)\log|z|+C_{2,\varepsilon}.
\end{aligned}
$$

Therefore by taking $k_\varepsilon\in\mathbb{N}$ and R_ε larger as necessary, we can ensure that as $\forall k\geq k_\varepsilon$ and for every z such that $|z|\geq 2R_\varepsilon$, we have

$$\frac{1}{2\pi}\int_{\{|z-y|<\frac{|z|}{2}\}}\left(\log\frac{|y|}{|z-y|}\right)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)} < \frac{4\varepsilon}{5}\log|z| + C_{2,\varepsilon}, \quad (5.6.25)$$

for a suitable constant $C_{2,\varepsilon}$.

Finally we note

$$\log\frac{|y|}{|z-y|} \leq \log 2, \text{ for } y \in B_{1/\varepsilon_k}\setminus\left(B_{\frac{|z|}{2}}(0)\cup B_{\frac{|z|}{2}}(z)\right), \quad (5.6.26)$$

and by (5.6.16), (5.6.24), and (5.6.26) we conclude:

$$\begin{aligned}\xi_k(z) &\leq \left(4(1+\alpha)-\frac{\varepsilon}{5}\left(\frac{2}{\alpha+2}+4\right)\right)\log\frac{1}{|z|}+\frac{M_k}{2\pi}\log 2+C_{1,\varepsilon}+C_{2,\varepsilon}\\ &\leq (4(1+\alpha)-\varepsilon)\log\frac{1}{|z|}+C_\varepsilon,\end{aligned}$$

$\forall|z| \geq 2R_\varepsilon$ and $\forall k \geq k_\varepsilon$, with a suitable positive constant C_ε. □

Remark 5.6.53 By the proof of Lemma 5.6.52 it follows that,
(a)

$$\text{if } \alpha \in (0,+\infty)\setminus\mathbb{N} \text{ or } u_k(0) = \max_{B_1} u_k,$$

then (5.6.17) holds for the full sequence ξ_k, since each convergent subsequence of ξ_k admits the *same* limit function: $\xi(z) = \log\frac{1}{(1+\gamma_\alpha|z|^{2(\alpha+1)})^2}$.
(b) The assumption

$$|\nabla V_k| \leq A \text{ in } B_1 \quad (5.6.27)$$

has entered to guarantee (together with the other hypothesis) that

$$\int_{B_1}|z|^{2\alpha_k}V_k e^{u_k} \to 8\pi(1+\alpha), \text{ as } k \to +\infty. \quad (5.6.28)$$

Therefore, in Lemma 5.6.52 we can do without (5.6.27) whenever (5.6.28) is otherwise available.

We are going to use (5.6.17) in order to improve the estimates above as follows.

Lemma 5.6.54 *Along a subsequence (denoted the same way) and for k sufficiently large, we have*

$$\left|\xi_k(z)+\frac{M_k}{2\pi}\log|z|\right| = O(1), \textit{ for } 3 \leq |z| \leq \frac{1}{\varepsilon_k}, \quad (5.6.29)$$

$$\left|\nabla\xi_k(z)+\frac{M_k}{2\pi}\frac{z}{|z|^2}\right| = 0\left(\frac{1}{|z|^2}\right), 3 \leq |z| \leq \frac{r_0}{\varepsilon_k}, \quad (5.6.30)$$

for given $r_0 \in (0,1)$ *and* M_k *as defined in (5.6.18).*

Proof. We are going to consider the subsequence for which (5.6.11) holds. We have:

$$\xi_k(z) + \frac{M_k}{2\pi}\log|z| = \frac{1}{2\pi}\int\limits_{B_{1/\varepsilon_k}}\left(\log\frac{|y||z|}{|y-z|}\right)|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k} + \psi_k, \quad (5.6.31)$$

with ψ_k uniformly bounded in $\bar{B}_{1/\varepsilon_k}$.

To estimate the integral in (5.6.31) we refer to Lemma 5.6.52, and find

$$|y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)} \le \frac{C}{1+|y|^3}, \quad (5.6.32)$$

$\forall y \in \bar{B}_{1/\varepsilon_k}$ and a suitable constant C, provided that, k is sufficiently large. Thus, we derive

$$\int\limits_{B_{1/\varepsilon_k}} |\log|y|||y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)} \le C, \quad (5.6.33)$$

for every $k \in \mathbb{N}$ and suitable $C > 0$.

Claim : For $|z| \ge 3$,

$$\left|\log\frac{|y||z|}{|y-z|}\right| \le 2|\log|y|| + \left(\log\frac{1}{|y-z|}\right)^+ + \log 2 \quad (5.6.34)$$

in $B_{1/\varepsilon_k}\setminus\{z\}$, where as usual, $f^+ = \max\{f, 0\}$ denotes the positive part of a given function f.

To verify (5.6.34), first let $y \in B_{\frac{|z|}{2}}(0)$. Then $|y-z| \ge \frac{|z|}{2} > 1$ and so $\left(\log\frac{1}{|y-z|}\right)^+ = 0$. Furthermore,

$$\left|\log\frac{|y||z|}{|y-z|}\right| = \left|\log|y| + \log\left|\frac{z}{|z|} - \frac{y}{|z|}\right|\right| \le \log|y| + \log 2$$

and (5.6.34) is verified in this case.

Now take $y : |y-z| < \frac{|z|}{2}$. Hence $|y| \ge \frac{|z|}{2} > 1$ and $\log\frac{|y||z|}{|y-z|} \ge \log 2|y| \ge \log|z| > 0$. Therefore,

$$\begin{aligned}\left|\log\frac{|y||z|}{|y-z|}\right| &= \log|y||z| + \log\frac{1}{|y-z|} \le 2\log|y| + \log 2 + \left(\log\frac{1}{|z-y|}\right)^+ \\ &= 2|\log|y|| + \log 2 + \left(\log\frac{1}{|z-y|}\right)^+.\end{aligned}$$

Finally, if $y \in B_{1/\varepsilon_k}\setminus\left(B_{\frac{|z|}{2}}(0)\cup B_{\frac{|z|}{2}}(z)\right)$, then $\left(\log\frac{1}{|z-y|}\right)^+ = 0$ while

$$\log\frac{|y||z|}{|y-z|} \ge \log\frac{|y||z|}{|y|+|z|} \ge \log\frac{|z|}{3} > 0.$$

Consequently, $\left|\log \frac{|y||z|}{|y-z|}\right| = \log \frac{|y||z|}{|y-z|} \le \log 2|y|$ and (5.6.34) follows also in this case.

By means of (5.6.34) we can immediately get (5.6.29) in view of (5.6.31). And we obtain

$$\begin{aligned}\left|\xi_k(z) + \frac{M_k}{2\pi}\log|z|\right| &\le \frac{1}{2\pi}\int_{B_{1/\varepsilon_k}} \left|\log\frac{|y||z|}{|z-y|}\right| |y|^{2\alpha_k} V_k(\varepsilon_k y) e^{\xi_k} + O(1) \\ &\le \frac{1}{\pi}\int_{B_{1/\varepsilon_k}} |\log|y|||y|^{2\alpha_k} V_k(\varepsilon_k y)e^{\xi_k} + \frac{M_k}{2\pi}\log 2 \\ &\quad + C\left(\int_{\{|z-y|<1\}} \log\frac{1}{|z-y|}\right) + O(1) = O(1).\end{aligned}$$

To obtain (5.6.30), we take advantage of (5.6.29) just established to deduce:

$$\xi_k(z) \le -\frac{M_k}{2\pi}\log(1+|z|) + C, \quad \forall z \in B_{1/\varepsilon_k}. \tag{5.6.35}$$

Since

$$\left|\nabla\xi_k(z) + \frac{M_k}{2\pi}\frac{z}{|z|^2}\right| \le \frac{1}{2\pi}\int_{B_{1/\varepsilon_k}} \left|\frac{z-y}{|z-y|^2} - \frac{z}{|z|^2}\right| |y|^{2\alpha_k} V_k(\varepsilon_k y)e^{\xi_k(y)} + O(\varepsilon_k) \tag{5.6.36}$$

in B_{r_0/ε_k}, then by taking k sufficiently large, we can guarantee that

$$\frac{M_k}{2\pi} - 2(\alpha_k+1) \ge \frac{3}{2}. \tag{5.6.37}$$

We proceed to estimate the integral in (5.6.36) in the three regions $D_1 = B_{\frac{|z|}{2}}(0)$, $D_2 = B_{\frac{|z|}{2}}(z)$ and $B_k' = B_{1/\varepsilon_k} \setminus (D_1 \cup D_2)$.

Using (5.6.35) and (5.6.37), for k large, we have

$$\begin{aligned}&\int_{D_1} \left|\frac{z-y}{|z-y|^2} - \frac{z}{|z|^2}\right| |y|^{2\alpha_k} V_k(\varepsilon_k y)e^{\xi_k(y)} = \int_{D_1} \frac{|y|}{|y-z||z|}|y|^{2\alpha_k} V_k(\varepsilon_k y)e^{\xi_k(y)} \\ &\le \frac{C}{|z|^2}\int_{\{|y|<|z|/2\}} \frac{|y|^{1+2\alpha_k}}{(1+|y|)^{M_k/2\pi}} \le \frac{C}{|z|^2}\int_0^{|z|/2} \frac{dr}{(1+r)^{3/2}} \le \frac{C}{|z|^2}.\end{aligned} \tag{5.6.38}$$

Since $D_2 \subset \{\frac{|z|}{2} \le |y| \le \frac{3}{2}|z|\}$, for sufficiently large k and $|z| \ge 1$, we find:

$$\begin{aligned}&\int_{D_2} \left|\frac{z-y}{|z-y|^2} - \frac{z}{|z|^2}\right| |y|^{2\alpha_k} V_k(\varepsilon_k y)e^{\xi_k(y)} \le C\int_{D_2} \frac{1}{|y-z|}|y|^{2\alpha_k - \frac{M_k}{2\pi}} \\ &\le C|z|^{2\alpha_k - \frac{M_k}{2\pi}} \int_{\{|z-y|<\frac{|z|}{2}\}} \frac{1}{|y-z|} \le C|z|^{2\alpha_k+1-\frac{M_k}{2\pi}} \le \frac{C}{|z|^{5/2}}.\end{aligned} \tag{5.6.39}$$

Finally, noting that

$$\left|\frac{z-y}{|z-y|^2}-\frac{z}{|z|^2}\right|=\frac{|y|}{|z-y||z|}\leq\frac{4}{|z|},\forall y\in B_k',\tag{5.6.40}$$

we conclude

$$\int_{B_k'}\left|\frac{z-y}{|z-y|^2}-\frac{z}{|z|^2}\right||y|^{2\alpha_k}V_k(\varepsilon_k y)e^{\xi_k(y)}\leq\frac{C}{|z|}\int_{\{|y|\geq\frac{|z|}{2}\}}|y|^{2\alpha_k-\frac{M_k}{2\pi}}\leq\frac{C}{|z|^{5/2}}\tag{5.6.41}$$

(provided) k is sufficiently large and $|z|\geq 1$. The estimate (5.6.30) follows immediately from (5.6.36), (5.6.38), (5.6.39), and (5.6.41). □

Observe that the estimates (5.6.29) and (5.6.30) can be recasted in terms of u_k, respectively, as follows:

$$u_k(z)=-\frac{M_k}{2\pi}\log|z|+\left(2(1+\alpha_k)-\frac{M_k}{2\pi}\right)\log\frac{1}{\varepsilon_k}+O(1),\tag{5.6.42}$$

for $3\varepsilon_k\leq|z|\leq 1$, and k sufficiently large;

$$\nabla u_k(z)=-\frac{M_k}{2\pi}\frac{z}{|z|^2}+O\left(\frac{\varepsilon_k}{|z|^2}\right)\tag{5.6.43}$$

for $3\varepsilon_k\leq|z|\leq r_0$, where $r_0\in(0,1)$ and k is sufficiently large.

Estimates (5.6.42) and (5.6.43) are crucial to establish the following important property:

Lemma 5.6.55 *Along a subsequence the following holds,*

$$\left|\frac{M_k}{2\pi}+4(\alpha_k+1)\right|=O\left((\log 1/\varepsilon_k)^{-1}\right),\ as\ k\to+\infty.\tag{5.6.44}$$

Proof. To obtain (5.6.44), we are going to use Pohozaev's identity (5.2.16) in the ball $\tilde{B}_k=\{|y|\leq\varepsilon_k\log\frac{1}{\varepsilon_k}\}$. This implies that

$$\begin{aligned}\int_{\partial\tilde{B}_k}r\left[\left(\frac{\partial u_k}{\partial\nu}\right)^2-\frac{1}{2}|\nabla u_k|^2\right]d\sigma&+\int_{\partial\tilde{B}_k}r^{1+2\alpha}V_ke^{u_k}\\&=\int_{\tilde{B}_k}2|y|^{2\alpha_k}V_ke^{u_k}+y\cdot\nabla\left(|y|^{2\alpha_k}V_k\right)e^{u_k}\\&=2(1+\alpha_k)\int_{\tilde{B}_k}|y|^{2\alpha_k}V_ke^{u_k}+\int_{\tilde{B}_k}|y|^{2\alpha_k}\left(y\cdot\nabla V_k\right)e^{u_k},\end{aligned}\tag{5.6.45}$$

for $r=|y|$.

On $\partial \tilde{B}_k$ we can use (5.6.43) to derive:

$$\int_{\partial \tilde{B}_k} r\left[\left(\frac{\partial u_k}{\partial \nu}\right)^2 - \frac{1}{2}|\nabla u_k|^2\right] d\sigma = \int_{\partial \tilde{B}_k} r\left[\frac{1}{2}\left(\frac{M_k}{2\pi}\right)^2 \frac{1}{r^2} + O\left(\frac{1}{\varepsilon_k^2(\log\frac{1}{\varepsilon_k})^3}\right)\right] d\sigma$$
$$= \frac{M_k^2}{4\pi} + O\left(\left(\log\frac{1}{\varepsilon_k}\right)^{-1}\right). \tag{5.6.46}$$

While (5.6.42) implies that

$$\int_{\partial \tilde{B}_k} r^{1+2\alpha_k} V_k e^{u_k} d\sigma = O\left(\left(\log\frac{1}{\varepsilon_k}\right)^{2(\alpha_k+1)-\frac{M_k}{2\pi}}\right),$$

$$\int_{\tilde{B}_k} |y|^{2\alpha_k} V_k e^{u_k} = M_k + \int_{B_1\setminus\tilde{B}_k} |y|^{2\alpha_k} V_k e^{u_k} = M_k + O\left((\log\frac{1}{\varepsilon_k})^{2(\alpha_k+1)-\frac{M_k}{2\pi}}\right),$$

and

$$\left|\int_{\tilde{B}_k} |y|^{2\alpha_k} y\cdot\nabla V_k e^{u_k}\right| \le O\left((\log\frac{1}{\varepsilon_k})^{2(\alpha_k+1)-\frac{M_k}{2\pi}}\right).$$

Therefore, in view of (5.6.37), we can use the estimates above to conclude

$$\frac{M_k^2}{4\pi} + O\left((\log\frac{1}{\varepsilon_k})^{-1}\right) = 2(\alpha_k+1)M_k + O\left((\log\frac{1}{\varepsilon_k})^{-1}\right)$$

provided that k is sufficiently large. Thus (5.6.44) follows immediately. □

Proof of Theorem 5.6.51. Using (5.6.11) and (5.6.12) together with (5.6.29) and (5.6.44) is sufficient for concluding:

$$\left|\xi_k(z) - \log\frac{1}{(1+\gamma_\alpha V_k(0)|z|^{2(\alpha_k+1)})^2}\right| \le C \text{ in } \bar{B}_{1/\varepsilon_k}.$$

Consequently, (5.6.6) follows by means of (5.6.7) and (5.6.8). □

As a direct consequences of Theorem 5.6.51, we obtain:

Corollary 5.6.56 *Let u_k satisfy (5.6.4) with $\alpha_k \to \alpha \in (0,+\infty)\setminus\mathbb{N}$ and V_k as in (5.5.6). Then the sequence u_k satisfies the pointwise estimate (5.6.6) in $\bar{B}_1$.*

Proof. It suffices to use Proposition 5.6.50 together with Corollary 5.4.24 to verify (5.6.5). Then by recalling Remark 5.6.53, we can guarantee the validity of (5.6.6) for the whole sequence u_k. □

Corollary 5.6.57 *Let u_k satisfy (5.6.4) with $\alpha_k = 0$ and V_k as in (5.5.6). For every $r \in (0, 1)$, there exists $C_r > 0$ such that*

$$\left| u_k(z) - \log \frac{\lambda_k}{(1 + \frac{V_k(z_k)}{8}\lambda_k |z - z_k|^2)^2} \right| \le C_r, \text{ in } \bar{B}_r, \tag{5.6.47}$$

where

$$z_k \to 0 : \ u_k(z_k) = \max_{\bar{B}_1} u_k \text{ and } \lambda_k = e^{u_k(z_k)}. \tag{5.6.48}$$

Proof. For a given $r \in (0, 1)$, let $\tilde{u}_k(z) = u_k(z_k + rz) + 2\log r$. We easily check that the sequence $\tilde{u}_k$ satisfies (5.6.4), (5.6.5) with $\alpha_k = 0$ and $\tilde{V}_k(z) = V_k(z_k + rz)$.

Furthermore, given $|z_1| = |z_2| = 1$, we use (5.6.14) and find

$$|\tilde{u}_k(z_1) - \tilde{u}_k(z_2)| \le \frac{1}{2\pi} \int_{B_1} \left| \log \frac{|z_k + rz_2 - y|}{|z_k + rz_1 - y|} \right| V_k(y) e^{u_k(y)} + O(1),$$

and

$$\int_{\{|y|<1\}} \left| \log \frac{|z_k + rz_2 - y|}{|z_k + rz_1 - y|} \right| V_k(y) e^{u_k(y)} \to 0, \text{ as } k \to +\infty.$$

Therefore, $\tilde{u}_k$ satisfies (5.6.1), as well as, all the other assumptions of Theorem 5.6.51. Since $\tilde{u}_k(0) = \max_{\bar{B}_1} \tilde{u}_k$, we can also use Remark 5.6.53 to conclude the validity of the pointwise estimate (5.6.6) for the whole sequence $\tilde{u}_k$.

That is,

$$\left| \tilde{u}_k(z) - \log \frac{e^{\tilde{u}_k(0)}}{\left(1 + \frac{\tilde{V}_k(0)}{8} e^{\tilde{u}_k(0)} |z|^2\right)^2} \right| \le C \text{ in } \bar{B}_1. \tag{5.6.49}$$

From (5.6.49) and the definition of $\tilde{u}_k$, we immediately derive (5.6.47). □

Remark 5.6.58 At this point, it is natural to *conjecture* that, when $\alpha = N \in \mathbb{N}$, the following estimate should hold for a solution sequence u_k, of (5.6.4):

$$\left| u_k(z) - \log \frac{\lambda_k}{\left(1 + \frac{V_k(z_k)}{8(N+1)^2} \lambda_k |z^{N+1} - z_k^{N+1}|^2\right)^2} \right| \le C \text{ in } \bar{B}_1, \tag{5.6.50}$$

with z_k and λ_k as defined in (5.6.48).

In support of (5.6.50), notice that in case:

$$\lambda_k |z_k|^{N+1} = O(1); \tag{5.6.51}$$

that is,

$$\frac{|z_k|}{\varepsilon_k} = O(1), \text{ with } \varepsilon_k = e^{-\frac{u_k(z_k)}{2(N+1)}},$$

then we can argue as in Corollary 5.4.24 and verify that (5.6.5) is satisfied. Then we may apply (5.6.6) together with (5.6.51) to obtain (5.6.50).

Thus, the truly delicate case to consider occurs when

$$\left|\frac{z_k}{\varepsilon_k}\right| \to +\infty. \tag{5.6.52}$$

In this situation, we can only give a partial contribution towards (5.6.50) by proving, in the following section, the validity of the stronger version of the "sup+inf" estimate as given in (5.5.61).

5.6.3 The inf + sup estimates revised

For given $c_0 > 0$, consider the problem:

$$\begin{cases} -\Delta u = |z|^{2\alpha} V e^u \text{ in } B_1, \\ \int_{B_1} |z|^{2\alpha} V e^u \le c_0, \\ \max_{\partial B_1} u - \min_{\partial B_1} u \le c_0. \end{cases} \tag{5.6.53}$$

The following holds:

Theorem 5.6.59 *If u satisfies (5.6.53) and if (5.5.43) holds, then for every $r \in (0, 1)$ there exists a constant $C > 0$ depending only on r, α, b_1, b_2, A, and c_0 such that*

$$\sup_{B_r} u + \inf_{B_1} u \le C. \tag{5.6.54}$$

Theorem 5.6.59 should be compared with Corollary 5.5.42. It has the advantage of relying upon assumptions that are often available in the applications.

Proof. To obtain (5.6.54) we argue by contradiction, and similarly to Corollary 5.5.42, we assume that there exists a sequence u_k, satisfying:

$$\begin{cases} -\Delta u_k = |z|^{2\alpha} V_k e^{u_k} \text{in } B_1, \\ \int_{B_1} |z|^{2\alpha} V_k e^{u_k} \le c_0, \\ \max_{\partial B_1} u_k - \min_{\partial B_1} u_k \le c_0, \end{cases} \tag{5.6.55}$$

with V_k satisfying (5.5.7) and such that for a sequence $\{z_k\} \subset B_1$ we have:

$$u_k(z_k) + \inf_{B_1} u_k \to +\infty \tag{5.6.56}$$

and (by virtue of Corollary 5.5.7)

$$z_k \to 0 \tag{5.6.57}$$

as $k \to +\infty$.

Without loss of generality, we can further assume

$$u_k(z_k) = \max_{\bar{B}_1} u_k, \tag{5.6.58}$$

$$|z|^{2\alpha} V_k e^{u_k} \to \beta\delta_{z=0}, \tag{5.6.59}$$

weakly in the sense of measure in B_1.

Thus, we can apply Theorem 5.4.28 and conclude that (5.6.59) holds with

$$\beta = 8\pi(1+\alpha). \tag{5.6.60}$$

Set $\varepsilon_k = e^{-\frac{u_k(z_k)}{2(\alpha+1)}}$. As already mentioned above, we only have to consider the case where

$$\left|\frac{z_k}{\varepsilon_k}\right| \to +\infty, \text{ as } k \to +\infty; \tag{5.6.61}$$

otherwise we could verify (5.6.5) and use (5.6.6) to contradict (5.6.56). Also notice that (5.6.61) is equivalent to,

$$u_k(z_k) + 2(\alpha+1)\log|z_k| \to +\infty, \text{ as } k \to +\infty. \tag{5.6.62}$$

To account for (5.6.62), we consider

$$v_k(z) = u_k(|z_k|z) + 2(\alpha+1)\log|z_k|$$

defined in $B_k := \{z : |z| \le \frac{2}{|z_k|}\}$. We have:

$$\int_{B_k} |z|^{2\alpha} V_k(|z_k|z) e^{v_k} = 8\pi(1+\alpha) + o(1), \tag{5.6.63}$$

as a consequence of (5.6.59) and (5.6.60).

Furthermore, (5.6.62) implies that,

$$v_k\left(\frac{z_k}{|z_k|}\right) \to +\infty, \text{ as } k \to +\infty;$$

that is (along a subsequence), v_k admits a blow-up point z_0 in the unit circle.

Claim: If z_0 is a blow-up point for v_k, then there exists $\rho_0 > 0$ sufficiently small and $C > 0$ such that

$$\max_{\partial B_{\rho_0}(z_0)} v_k - \min_{\partial B_{\rho_0}(z_0)} v_k \le C. \tag{5.6.64}$$

To establish the claim we use Green's representation formula as in (5.6.14) and for $r_k = |z_k|$ we write

$$v_k(x) - v_k(\tilde{x}) = \frac{1}{2\pi} \int_{B_k} \log \frac{|\tilde{x} - y|}{|x - y|} |y|^{2\alpha} V_k(r_k y) e^{v_k} + O(1), \tag{5.6.65}$$

$\forall x, \tilde{x} \in \Omega_k$. Let $\rho_0 \in (0, 1)$ sufficiently small, so that z_0 is the only blow-up point for v_k in $\overline{B_{2\rho_0}(z_0)} \subset B_k$ and let $D_0 = \{y \in B_k : \frac{\rho_0}{2} \le |y - z_0| \le 2\rho_0\}$. It is easy to check that for $x, \tilde{x} \in \partial B_{\rho_0}(z_0)$

$$\sup_{B_k \setminus D_0} \left| \log \frac{|\tilde{x} - y|}{|x - y|} \right| \le C_1, \tag{5.6.66}$$

while

$$\sup_{D_0} v_k \le C_2, \tag{5.6.67}$$

for suitable positive constants C_1 and C_2. Thus, for every $x, \tilde{x} \in \partial B_{\rho_0}(z_0)$, from (5.6.65) we can estimate:

$$\begin{aligned}
|v_k(x) - v_k(\tilde{x})| &\le \frac{1}{2\pi} \int_{D_0} \left| \log \frac{|\tilde{x} - y|}{|x - y|} \right| |y|^{2\alpha} V_k(r_k y) e^{v_k} \\
&\quad + \frac{1}{2\pi} \int_{\Omega_k \setminus D_0} \left| \log \frac{|\tilde{x} - y|}{|x - y|} \right| |y|^{2\alpha} V_k(r_k y) e^{v_k} \\
&\quad + O(1) \le C_3 \int_{B_{3\rho_0}(0)} |\log |z|| + C_4 \int_{\Omega_k} |y|^{2\alpha} V_k(r_k z) e^{v_k} + O(1) \le C
\end{aligned}$$

and the Claim follows. Thus, we can use Corollary 5.6.57 for v_k around $\frac{z_k}{|z_k|}$ and derive

$$\left| v_k(z) - \log \frac{\mu_k}{(1 + \frac{1}{8} V_k(z_k) \mu_k |z - \frac{z_k}{|z_k|}|^2)^2} \right| \le C \tag{5.6.68}$$

in $\overline{B_{\rho_0}(z_0)}$, with $\mu_k = e^{v_k\left(\frac{z_k}{|z_k|}\right)}$.

As a consequence of (5.6.68) we find

$$\int_{B_\delta(z_0)} |z|^{2\alpha} V_k(|z_k| z) e^{v_k} = 8\pi + o(1), \text{ as } k \to +\infty, \tag{5.6.69}$$

$\forall \delta \in (0, \rho_0)$, and

$$v_k\left(\frac{z_k}{|z_k|}\right) + \inf_{\partial B_{\rho_0}(z_0)} v_k \quad \leq C \tag{5.6.70}$$

(note that (5.6.69) also follows by Theorem 5.4.28 applied with $\alpha = 0$).

By (5.6.63) and (5.6.69), we see that zero cannot be a blow-up point for v_k. Indeed, if this was the case, then we could use the Claim above with $z_0 = 0$, and by Theorem 5.4.28 conclude that

$$\int_{B_\varepsilon(0)} |z|^{2\alpha} V_k(|z_k|z)e^{v_k} = 8\pi(1+\alpha) + o(1),$$

for every $\varepsilon > 0$ sufficiently small, in contradiction with (5.6.63) and (5.6.69).

Hence for small $\varepsilon_0 > 0$, necessarily:

$$\max_{\bar{B}_{2\varepsilon_0}} v_k \to -\infty, \text{ as } k \to +\infty. \tag{5.6.71}$$

Thus, we readily check that alternative (a) of Theorem 5.5.36 applies to v_k and implies:

$$\inf_{B_1} u_k \leq \max_{\{|z|\leq\varepsilon_0\}} v_k + 2(\alpha+1)\log|z_k| + C. \tag{5.6.72}$$

Furthermore, using (5.6.71) and (5.6.65), we can argue, as in the proof of the Claim above, to find a constant $C > 0$ such that

$$|v_k(x) - v_k(\tilde{x})| \leq C, \ \forall x \in B_{\varepsilon_0} \text{ and } \tilde{x} \in \partial B_{\rho_0}(z_0).$$

In other words,

$$\max_{\bar{B}_{\varepsilon_0}} v_k \leq \min_{\partial B_{\rho_0}(z_0)} v_k + C. \tag{5.6.73}$$

Consequently, from (5.6.72) and (5.6.73), we deduce the following estimate

$$u_k(z_k) + \inf_{B_1} u_k \leq \max_{\bar{B}_{\varepsilon_0}} v_k + v_k\left(\frac{z_k}{|z_k|}\right) + C \leq \min_{\partial B_{\rho_0}(z_0)} v_k + v_k\left(\frac{z_k}{|z_k|}\right) + C.$$

Therefore, we can use (5.6.70) to obtain a contradiction to (5.6.56), and conclude the proof. □

As a matter of fact, the arguments above allow us to deduce a (global) "inf+sup" estimate in the same spirit of Corollary 5.5.40, relative to solutions of the problem:

$$\begin{cases} -\Delta u = W(z)e^u \text{ in } \Omega, \\ \int_\Omega W(z)e^u \leq c_0, \\ \sup_{\partial\Omega} - \inf_{\partial\Omega} \leq c_0, \end{cases} \tag{5.6.74}$$

for given $c_0 > 0$, where we assume that

$$W(z) = \prod_{i=1}^{m} |z - z_i|^{2\alpha_i} V(z) \tag{5.6.75}$$

with

$$\{z_1, \dots, z_m\} \subset \Omega \text{ distinct points and } \alpha_i \geq 0 \text{ for every } i = 1, \dots, m, \tag{5.6.76}$$

and

$$V \in C^{0,1}(\Omega) \text{ satisfies } 0 < b_1 \leq V \leq b_2 \text{ and } |\nabla V| \leq A. \tag{5.6.77}$$

Theorem 5.6.60 *Every solution u of (5.6.74) for which (5.6.75)–(5.6.77) hold, satisfies*

$$\sup_K u + \inf_\Omega u \leq C,$$

for every compact set $K \subset\subset \Omega$, with C depending only on the given data c_0, α_i $i = 1, \dots, m$, b_1, b_2, A, and dist$(K, \partial\Omega)$.

5.7 The concentration-compactnesss principle completed

We conclude this Chapter by stating appropriate formulations of the results established above useful in the sequel. Firstly, we complete the "concentration" result in Theorem 5.4.34 as follows.

Theorem 5.7.61 *Under the assumptions of Theorem 5.4.34 suppose in addition that*

$$V_k \in C^{0,1}(\Omega) : \ |\nabla V_k| \leq A \text{ in } \Omega. \tag{5.7.1}$$

Then property (c) (iii) holds with

$$\beta_j \in \begin{cases} 8\pi\mathbb{N} & \text{if } q_j \notin \{z_1, \dots z_m\}, \\ 8\pi\left(\mathbb{N} + \alpha_i\right) \cup 8\pi\mathbb{N} & \text{if } q_j = z_i \text{ for some } i \in \{1, \dots, m\}. \end{cases} \tag{5.7.2}$$

Furthermore, if

$$\sup_{\partial\Omega} u_k - \inf_{\partial\Omega} u_k \leq C,$$

then the values β_j can be further specified as follows:

$$\beta_j = \begin{cases} 8\pi & \text{if } q_j \notin \{z_1, \dots z_m\} \\ 8\pi(1 + \alpha_i) & \text{if } q_j = z_i \text{ for some } i \in \{1, \dots, m\}. \end{cases} \tag{5.7.3}$$

Proof. Given a blow-up point $q \in S$, then for a (subsequence of) u_k and for $r_0 > 0$ sufficiently small, we consider the sequence $\tilde{u}_k(z) = u_k(q + r_0 z) + 2(\alpha_k + 1)\log r_0$, defined for $z \in B_1$, where we take $\alpha_k = 0$ if $q \notin \{z_1, \dots, z_m\}$ or $\alpha_k = \alpha_{i,k}$ if $q = z_i$ for some $i = 1, \dots, m$. Hence (5.7.1) follows simply by applying Theorem 5.5.44 to $\tilde{u}_k$. If case (5.7.3) also holds, then according to (5.6.64) we can assume:

$$\sup_{\partial B_{r_0}(q)} u_k - \inf_{\partial B_{r_0}(q)} u_k \leq C,$$

for a suitable $C > 0$. Therefore, in this case, (5.7.3) follows by applying Theorem 5.4.28 to the sequence $\tilde{u}_k$. □

We go back to analyzing our original problem (5.1.1), where we consider solution-sequences of Liouville equations in the Mean–Field form.

In fact, we discuss a more general framework, where u_k is defined over a closed surface (M, g). Denote with Δ_g, ∇_g and $d\sigma_g$ respectively the Laplace–Beltrami operator, the gradient and the volume element relative to the metric g on M. Also, let d_g denote the distance function on (M, g). Let w_k satisfy

$$\begin{cases} -\Delta_g w_k = \mu_k \left(\frac{h_k(x) e^{w_k}}{\int_M h_k(x) e^{w_k}\, d\sigma_g} - \frac{1}{|M|} \right) \text{ in } M, \\ w_k \in H^1(M) : \int_M w_k d\sigma_g = 0, \end{cases} \tag{5.7.4}$$

and for the moment, we assume that

$$\mu_k \to \mu > 0 \tag{5.7.5}$$

$$h_k = e^{u_{0,k}} \in C(M) : \ u_{0,k} \in \mathrm{L}^1(M) \text{ and } \int_M u_{0,k}\, d\sigma_g = 0. \tag{5.7.6}$$

We start by observing the following useful fact:

Lemma 5.7.62 *Let w_k satisfies (5.7.4) and assume (5.7.5) and (5.7.6) hold. Then w_k is uniformly bounded in $W^{1,q}(M)$ $\forall 1 \leq q < 2$.*

Proof. The argument is analogous to that of Lemma 4.1.2 . Hence, let p, the dual exponent of q (that is $\frac{1}{q} + \frac{1}{p} = 1$), so that $p > 2$; and let us consider

$$\varphi \in W^{1,p}(M) \text{ such that } ||\nabla_g \varphi||_{\mathrm{L}^p(M)} = 1, \ \int_M \varphi\, d\sigma_g = 0. \tag{5.7.7}$$

By the Sobolev embedding theorem (cf. [Au]), we know that $\varphi \in C^0(M)$ and $||\varphi||_{\mathrm{L}^\infty(M)} \leq c_0$ for a suitable constannt $c_0 > 0$. By taking into account (5.7.5), we obtain

$$\int_M \nabla_g w_k \nabla_g \varphi = \mu_k \int_M \frac{h_k(x) e^{w_k}}{\int_M h_k(x) e^{w_k}\, d\sigma_g} \varphi \leq \mu_k ||\varphi||_{\mathrm{L}^\infty(M)} \leq C$$

with $C > 0$ a suitable constant. Therefore,

$$||\nabla_g w_k||^q_{L^q(M)} = \sup\{\int_M \nabla_g w_k \cdot \nabla\varphi,\ \varphi \text{ satisying (5.7.7)}\} \leq C \qquad (5.7.8)$$

as claimed. □

Clearly, we can extend the notion of the blow-up point, $p \in M$ for the sequence w_k in M, simply by asking if

$$\exists\{z_k\} \subset M :\ z_k \to p \text{ and } w_k(z_k) \to +\infty,\ \text{as } k \to +\infty. \qquad (5.7.9)$$

Remark 5.7.63 If the sequence w_k satisfies (5.7.4), then we can use standard elliptic regularity theory, to see that (5.7.9) is equivalent to the property:

$$w_k(z_k) - \log \int_M h_k e^{w_k} \to +\infty,\ \text{as } k \to +\infty. \qquad (5.7.10)$$

Proposition 5.7.64 *Let w_k satisfies (5.7.4) and assume that (5.7.5) and (5.7.6) hold. If $h_k \to h$ uniformly in M and p is a blow-up point for w_k such that $h(p) > 0$, then*

$$\liminf_{n\to\infty} \mu_k \frac{\int_{U_\delta(p)} h_k e^{w_k}}{\int_M h_k e^{w_k}} \geq 8\pi, \qquad (5.7.11)$$

where $U_\delta(p) = \{q \in M :\ d_g(p, q) < \delta\}$. Furthermore, if in (5.7.5) we have $\mu = 8\pi$, then p is the only blow-up point for w_k, and along a subsequence we have:

$$\frac{h_k e^{w_k}}{\int_M h_k e^{w_k}} \rightharpoonup \delta_p \text{ weakly in the sense of measure in } M, \qquad (5.7.12)$$

$$\sup_D \left(w_k - \log \int_M h_k e^{w_k}\, d\sigma_g \right) \to -\infty\ \forall D \subset\subset M \setminus \{p\}. \qquad (5.7.13)$$

Proof. In a small neighborhood of p in M, we define an isothermal coordinate system $y = (y_1, y_2)$ centered at the origin (i.e., $y(p) = 0$) such that

$$ds^2 = e^\varphi(dy_1^2 + dy_2^2) \text{ and } d\sigma_g = e^\varphi dy_1 dy_2,$$

with a smooth function $\varphi = \varphi(y)$ satisfying

$$\varphi(0) = 0,\ \nabla\varphi(0) = 0 \text{ and } -\Delta\varphi = 2Ke^\varphi \text{ in } B_{r_0}(y)$$

with K the Gauss curvature of M.

Without ambiguity, we write $w_k = w_k(y)$ to express w_k in such a coordinate system. Define

$$u_k(y) = w_k(y) - \log \int_M h_k e^{w_k}\, d\sigma_g - \frac{\mu_k |y|^2}{4|M|},\ y \in B_{r_0}(0). \qquad (5.7.14)$$

Hence u_k admits a blow-up point at the origin and satisfies

$$\begin{cases} -\Delta u_k = W_k e^{u_k} \text{ in } B_{r_0}(0), \\ \int_{B_{r_0}(0)} W_k e^{u_k} \leq C, \end{cases} \tag{5.7.15}$$

with $W_k = \mu_k h_k e^{\varphi} \to \mu h e^{\varphi} := W$, uniformly in $B_{r_0}(0)$, and with $W(0) > 0$.

Therefore, we can apply Proposition 5.4.20 to see that

$$\liminf_{n\to\infty} \int_{B_\delta(0)} W_k e^{u_k} \geq 8\pi, \ \forall \delta \in (0, r_0)$$

and in turn deduce (5.7.11). Furthermore, by means of property (c) of Proposition 5.4.32 (along a subsequence), we know also that

$$\sup_{\partial U_\delta(p)} \left(w_k - \log \int_M h_k e^{w_k}\, d\sigma_g \right) \to -\infty, \tag{5.7.16}$$

as $k \to +\infty$, and for $\delta > 0$ sufficiently small.

Next assume that $\mu_k \to 8\pi$. Then by (5.7.11) we see that necessarily

$$\frac{\int_{U_\delta(p)} h_k e^{w_k}\, d\sigma_g}{\int_M h_k e^{w_k}\, d\sigma_g} \to 1 \text{ as } k \to \infty, \ \forall \delta > 0, \tag{5.7.17}$$

or in other words,

$$\frac{\int_{M\setminus U_\delta(p)} h_k e^{w_k}\, d\sigma_g}{\int_M h_k e^{w_k}\, d\sigma_g} \to 0 \text{ as } k \to \infty.$$

Therefore, if as above we use isothermal coordinates to localize our problem around a point $q \in M \setminus \{p\}$, we obtain that in this case the sequence u_k, given in (5.7.14), satisfies (5.7.15) with the additional property:

$$\int_{B_{r_0}(0)} W_k e^{u_k} \to 0, \text{ as } k \to \infty, \tag{5.7.18}$$

for $r_0 > 0$ sufficiently small. From (5.7.14) we also see that $||u_k^+||_{L^1}$ is uniformly bounded, as we have:

$$u_k^+(y) \leq |w_k(y)| - \log \fint h_k e^{w_k}\, d\sigma_g + \log |M| \leq |w_k(y)| + \log |M|.$$

Since $\log \fint_M h_k e^{w_k} \geq 0$, and (by Lemma 5.7.62), we know that w_k is uniformly bounded in L^1-norm.

Thus, we are in a position to apply Proposition 5.3.13 to see that u_k^+ is uniformly bounded in $B_\delta(0)$ for every $\delta \in (0, r_0)$. At this point, recalling (5.7.16), we can use Corollary 5.2.9 and conclude

$$\sup_{B_\delta(0)} u_k \leq \beta \inf_{B_\delta(0)} u_k + C,$$

for $\beta \in (0,1)$ and $C > 0$ suitable constants. Notice that since we have made no assumptions about the size of the zero set of h_k, we cannot yet conclude that $\inf_{B_\delta} u_k \to -\infty$ as $k \to -\infty$ from (5.7.18). Nevertheless, by compactness, we can patch all such "local" information together to find that, for every $\delta > 0$ sufficiently small,

$$\sup_{M \setminus U_\delta(p)} \left(w_k - \log \int_M h_k e^{w_k} \, d\sigma_g \right) \le \beta \inf_{M \setminus U_\delta(p)} \left(w_k - \log \int_M h_k e^{w_k} \, d\sigma_g \right) + C$$

with suitable constants $\beta \in (0,1)$ and $C > 0$.

By virtue of (5.7.16) we check that

$$\inf_{M \setminus U_\delta(p)} \left(w_k - \log \int_M h_k e^{w_k} \, d\sigma_g \right) \to -\infty,$$

(along a subsequence) and (5.7.13) follows. Also we derive (5.7.12) by virtue of (5.7.17) □

To analyze the behavior of w_k around a general blow-up point, we focus on the case where

$$h_k(q) = \prod_{j=1}^{m} (d_g(q, p_j))^{\alpha_{j,k}} V_k \tag{5.7.19}$$

with $p_1, \ldots, p_m$ distinct points on M, and

$$0 \le \alpha_{j,k} \to \alpha_j \text{ as } k \to +\infty, \; j = 1, \ldots, m \tag{5.7.20}$$

$$V_k \in C^{0,1}(M) : \; 0 < a \le V_k \le b \text{ and } |\nabla_g V_k| \le A \text{ in } M \tag{5.7.21}$$

for suitable constants b_1, b_2 and $A > 0$.

By taking advantage of the analysis carried out in the previous sections, we obtain the following "concentration/compactness" result for w_k:

Theorem 5.7.65 *Let w_k satisfy (5.7.4) and assume that (5.7.5), (5.7.6), and (5.7.19)–(5.7.21) hold. Along a subsequence (denoted the same way), one of the following alternative holds:*
(i) w_k converges uniformly on M.
(ii) There exists a finite set of blow-up points $S = \{q_1, \ldots, q_s\} \subset M$ with the following properties:
(a) there exist $\{q_{j,k}\} \subset M$: $q_{j,k} \to q_j$ and $w_{j,k}(q_{j,k}) \to +\infty$, as $k \to +\infty$, $j = 1, \ldots, s$;
(b)

$$\max_D \left(w_k - \log \int_M h_k e^{w_k} \, d\sigma_g \right) \to -\infty, \; \forall D \subset\subset M \setminus S; \tag{5.7.22}$$

(c) as $k \to +\infty$,

$$\mu_k \frac{h_k e^{w_k}}{\int_M h_k e^{w_k}\, d\sigma_g} \to \sum_{j=1}^{s} \beta_j \delta_{q_j}, \text{ weakly in the sense of measure on } M, \tag{5.7.23}$$

and

$$\beta_j = \begin{cases} 8\pi & \text{if } q_j \notin \{p_1, \dots, p_m\} \\ 8\pi(1+\alpha_i) & \text{if } q_j = p_i \text{ for some } i \in \{1, \dots, m\}. \end{cases} \tag{5.7.24}$$

In particular, setting

$$\Gamma = 8\pi\mathbb{N} \cup \left\{ 8\pi \left(\mathbb{N} + \sum_{j \in J} \alpha_j \right) \text{ for every set } J \subseteq \{1, \dots, m\} \right\}, \tag{5.7.25}$$

alternative (ii) then holds if and only if μ in (5.7.5) satisfies: $\mu \in \Gamma$.

Theorem 5.7.65 extends a result of Li in [L2] concerning the case where the function h_k is uniformly bounded away from zero in M, i.e., in our notation when $\alpha_{j,k} = 0$ in (5.7.19) $\forall j = 1, \dots, m$ and $\forall k \in \mathbb{N}$.

Proof of Theorem 5.7.65. If $\max_M w_k \leq C$ for a suitable $C > 0$, then the right-hand side of (5.7.4) is uniformly bounded in $L^\infty(M)$. So by standard elliptic estimates and Sobolev's embeddings (cf. [Au]), we see that w_k is uniformly bounded in $C^{2,\gamma}(M)$ for $\gamma \in (0, 1)$, and we deduce (i). Hence assume that, along a subsequence, we have:

$$\max_M w_k \to +\infty, \text{ as } k \to \infty \tag{5.7.26}$$

or equivalently,

$$\max_M \left(w_k - \log \int_M h_k e^{w_k}\, d\sigma_g \right) \to +\infty, \text{ as } k \to \infty.$$

As above for $q \in M$, we use a local isothermal coordinate system $y = (y_1, y_2)$ centered at q, such that, for the sequence u_k in (5.7.14), we find

$$\begin{cases} -\Delta u_k = |y|^{2\alpha_k} \tilde{V}_k e^{u_k} \text{ in } B_{r_0}(0), \\ \int_{B_{r_0}(0)} |y|^{2\alpha_k} \tilde{V}_k e^{u_k} \leq C, \end{cases} \tag{5.7.27}$$

(for $r_0 > 0$ sufficiently small) with $\tilde{V}_k = \mu_k V_k e^{\varphi}$; while $\alpha_k = 0$ if $q \notin \{p_1, \dots, p_m\}$, or $\alpha_k = \alpha_{j,k}$ if $q = p_j$ for some $j \in \{1, \dots, m\}$.

According to our assumptions of (5.7.5) and (5.7.21), we can apply Proposition 5.4.32 to each of the local problems of (5.7.27). By (5.7.26) and a compactness argument, we conclude the existence of a finite set of blow-up points, $S = \{q_1, \dots, q_s\}$, such that properties (5.7.22) and (5.7.23) hold for the sequence w_k. Notice in particular that $\sup_D w_k \le C$, for every $D \subset\subset M \setminus S$. With this information, we can use elliptic estimates as above, to find that (along a subsequence) there holds:

$$w_k \to \sum_{j=1}^{s} \beta_j G(q, q_j) \text{ uniformly in } C^{2,\gamma}_{\text{loc}}(M \setminus S),$$

$\gamma \in (0, 1)$ and G the Green's function in (2.5.11). Thus, we check that w_k admits uniformly bounded oscillations in the boundary of a small neighborhood of $q_j \in S$, for every $j = 1, \dots, s$. At this point, we can use Theorem 5.4.28 for the corresponding localized sequence u_k to conclude (5.7.24). □

Remark 5.7.66 If M admits smooth boundary ∂M where we impose Dirichlet boundary conditions, then it is possible to rule out that a blow-up point occurs on ∂M (e.g., see [MW]). So, for a solution sequences w_k satisfying:

$$\begin{cases} -\Delta_g w_k = \mu_k \frac{h_k e^{w_k}}{\int_M h_k e^{w_k}} & \text{in } M \\ w_k = 0 & \text{on } \partial M, \end{cases} \tag{5.7.28}$$

the analysis above suffices to obtain a "concentration-compactness" result in complete analogy to that stated in Theorem 5.7.65. Notice in particular that M could be taken as a regular subdomain of $\mathbb{R}^2$.

If we require that Neumann boundary conditions apply on ∂M, then we face a more delicate situation as boundary blow-up occurs in this case. We refer to [WW1], [WW2], and references therein for a discussion of this situation.

The analysis carried out in this chapter follows the spirit of that developed for several other problems in geometry and physics where "concentration" phenomena occur naturally. In this framework one aims to identify the regime under which "concentrations" cannot develop. Or when "concentrations" occur, one aims to describe (as accurately as possible) the "bubble" profile that the concentrating-sequence assumes during blow-up.

Actually, it has been possible to use perturbation and gluing techniques to construct explicit classes of solutions which admit a "concentration" behavior with a prescribed "bubble" profile (as indicated by the blow-up analysis). In this respect and in relation to the aim of these notes, we mention the following contributions: [Re], [BC1], [BC2], [Ba], [KMPS], [NW], [Pa], [Sc1], [Sc2], [BP], [Es], [DDeM], [DeKM], [EGP], [Dr], [DHR], [AD], [Fl1], [AS], [PR], [We], and references therein; and mention that by no means this furnishes a complete list of references on the subject.

5.8 Final remarks and open problems

As usual we conclude with a discussion of some problems in blow-up analysis, which stem from the study of vortices and are left *open* by our investigation.

A first problem concerns the validity of an Harnack-type inequality in the following "inf + sup" form:

Open problem: Given $\alpha_0 > 0$, V satisfying (5.5.43) and $K \subset\subset B_1$ compact, is it true that for any $\alpha \in [0, \alpha_0]$ a solution of the equation

$$-\Delta u = |x|^{2\alpha} V e^u$$

satisfies

$$\sup_K u + \inf_{B_1} u \leq C$$

with $C > 0$ a constant depending only on α_0, dist $(K, \partial B_1)$ and the constants b_1, b_2, and A in (5.5.43)?

So far we know that the above property holds when $\alpha_0 = 0$, or when $\alpha_0 > 0$ but $K \subset\subset B_1 \setminus \{0\}$ (cf. [BLS]), or for any $\alpha_0 > 0$ but $K = \{0\}$ (cf. [T6]).

The general case has been handled in [BCLT], but only under some additional conditions as stated in Theorem 5.6.59 and Theorem 5.6.60 above.

Another aspect of our analysis which remains wide open for investigation concerns the validity of "concentration-compactness" principles for systems, and more specifically for the $SU(3)-$Toda system (4.5.3) or (4.5.11) and (4.5.35).

Any progress in this direction would have direct impact in the study of the asymptotic behavior of periodic $SU(3)-$vortices of the "non-topological-type" or "mixed-type", as given for instance in Theorem 4.5.35.

Towards this goal we recall the important contribution of Jost–Lin–Wang [JoLW], which takes advantage of the boundary conditions in order to handle the "regular" Toda-system, where the Dirac measures are neglected.

So the true delicate situation, yet to be understood, concerns the case where Dirac measures are involved and blow-up occurs at points where such measures are supported. Hence we pose the following

Open problem: Determine to what extent the (blow-up) analysis of section 5.4.2 remains valid when the single equation (5.4.1) is replaced, for instance, by an $SU(3)-$Toda system like the following:

$$\begin{cases} -\Delta v_{1,k} = 2|x|^{2\alpha_{1,k}} V_{1,k} e^{v_{1,k}} - |x - x_k|^{2\alpha_{2,k}} V_{2,k} e^{v_{2,k}} & \text{in } B_1 \\ -\Delta v_{2,k} = 2|x - x_k|^{2\alpha_{2,k}} V_{2,k} e^{v_{2,k}} - |x|^{2\alpha_{1,k}} V_{1,k} e^{v_{1,k}} & \text{in } B_1, \end{cases} \tag{5.8.1}$$

where $x_k \in B_1$ and both $V_{1,k}$ and $V_{2,k}$ satisfy assumptions of the (5.4.2) type and $\alpha_{1,k}$ and $\alpha_{2,k}$ satisfy (5.4.3).

The analysis of (5.8.1) should provide a first step towards the main goal that would be to answer the following:

Open question: In which form does a "concentration-compactness" principle of the type given by Theorem 5.4.34 (or Theorem 5.7.61), hold for a sequence of solutions of the $SU(3)-$Toda system of (4.5.3) (or more generally for systems of the (2.5.24) type) in the presence of Dirac measures?

Does the concentration phenomenon occur only according to certain "quantized" properties?

How does the presence of boundary conditions influence the answer to the above questions?

6

Mean Field Equations of Liouville-Type

6.1 Preliminaries

In this Chapter we show how to apply the analytical results established in Chapter 5 to the study of *Mean Field Equations* of the Liouville-type over a closed Riemann surface (M, g). More precisely, for a given $\mu > 0$ and $h \in L^\infty(M)$, we consider the mean field equation:

$$\begin{cases} -\Delta_g w = \mu \left(\frac{h(x)e^w}{\int_M h(x)e^w \, d\sigma_g} - \frac{1}{|M|} \right) \text{ in } M \\ \int_M w d\sigma_g = 0, \end{cases} \tag{6.1.1}$$

where we recall that Δ_g and $d\sigma_g$ denote, respectively, the Laplace–Beltrami operator and the volume element relative to the metric g on M. We shall be interested in analyzing (6.1.1) when h takes the form

$$h(p) = \prod_{j=1}^{m} \left(d_g(p, p_j)\right)^{2\alpha_j} V(p) \in C^\gamma(M), \ \gamma \in (0, 1] \text{ and } V > 0, \tag{6.1.2}$$

where $\{p_1, \ldots, p_m\} \subset M$ is a set of *distinct* points, $\alpha_j > 0$ for every $j = 1, \ldots, m$, and d_g denotes the distance function on (M, g).

Due to the structure of (6.1.1), we lose no generality in assuming that h is conveniently normalized as follows:

$$\int_M \log h \, d\sigma_g = 0. \tag{6.1.3}$$

We aim to obtain existence results for (6.1.1), as well as, compactness for the solution set, according to the value of the parameter $\mu > 0$.

As a direct consequence of Theorem 5.7.65, we can claim the following about the solutions of (6.1.1).

Proposition 6.1.1 *Let h satisfy (6.1.2) and (6.1.3) with*

$$V \in C^{0,1}(M): \ 0 < b_1 \le V \le b_2 \text{ and } |\nabla_g V| \le A \text{ in } M, \tag{6.1.4}$$

for given positive constants b_1, b_2, and A. For every compact set $\Lambda \subset \mathbb{R}^+ \setminus \Gamma$, there exists a constant $C > 0$, such that every solution w of (6.1.1) with $\mu \in \Lambda$ satisfies

$$||w||_{C^{2,\gamma}(M)} \le C, \ \gamma \in (0,1)$$

and C depends only on b_1, b_2, A, and $\max\{\mu : \mu \in \Lambda\}$.

As a consequence of Proposition 6.1.1, we know that for every $\mu \in \mathbb{R}^+ \setminus \Gamma$, the Leray–Schauder degree d_μ of the Fredholm map, $I + \mu T_h$, is well defined at zero. Recall that T_h is the compact operator defined in (2.5.6) which is associated to problem (6.1.1). In fact, d_μ is constant on the family of numerable open intervals I_n such that $\partial I_n \subset \Gamma$ and

$$\mathbb{R}^+ \setminus \Gamma = \cup_{n=1}^{+\infty} I_n. \tag{6.1.5}$$

It is an interesting open problem to compute explicitly the value of d_μ in terms of the integer $n \in \mathbb{N}$. For $n = 1$, it is easy to check that $I_1 = (0, 8\pi)$ and $d_\mu = 1$ $\forall \mu \in (0, 8\pi)$ (see [L2]). On the other hand, for $n \ge 2$ an explicit formula for d_μ is available only when h *never* vanishes over M, namely (in our notation), when $\alpha_j = 0$ $\forall j = 1, \dots, m$. In this case, $\Gamma = 8\pi\mathbb{N}$ and $I_n = (8\pi(n-1), 8\pi n)$. As already mentioned, Chen–Lin in [ChL2] succeeded in expressing d_μ in terms of n and the Euler characteristics $\chi(M)$ of M for every $\mu \in I_n$; see (2.5.10).

An equivalent formula is not yet available for the case when h vanishes at some points on M, as in (6.1.2). In this case, we expect the topological role of M to be replaced with the punctured manifold $M \setminus \{p_1, \dots, p_m\}$. Progress in this direction has been made by Chen–Lin–Wang in [ChLW], but only for the second interval I_2 in (6.1.5). In fact, assuming that h satisfies (6.1.2) with $\alpha_j \ge 1$ for every $j = 1, \dots, m$, then $I_2 = (8\pi, 16\pi)$ and as shown in [ChLW]

$$d_\mu = \begin{cases} 1, & \forall\, \mu \in (0, 8\pi) \\ \chi(M) + m + 1, & \forall\, \mu \in (8\pi, 16\pi) \end{cases} \tag{6.1.6}$$

with m the number of distinct zeroes of h in M.

Notice that, if h never vanishes on M, then “formally” we can take $m = 0$ in (6.1.6) and see that it reduces to (2.5.10). For later use, we mention that both Proposition 6.1.1 and the degree formulae discussed above hold for an elliptic 2×2 system that plays a crucial role in the study of electroweak periodic vortices, as discussed in Chapter 7.

More precisely, we consider the system:

$$\begin{cases} -\Delta_g w_1 = \mu \left(\frac{he^{w_1}}{\int_M he^{w_1}\, d\sigma_g} - \frac{1}{|M|} \right) + \lambda \left(\frac{fe^{w_2}}{\int_M fe^{w_2}\, d\sigma_g} - \frac{1}{|M|} \right) \text{ in } M, \\ \Delta_g w_2 = \frac{\mu}{2} \left(\frac{he^{w_1}}{\int_M he^{w_1}\, d\sigma_g} - \frac{1}{|M|} \right) + \frac{\lambda}{2\cos^2\theta} \left(\frac{fe^{w_2}}{\int_M fe^{w_2}\, d\sigma_g} - \frac{1}{|M|} \right) \text{ in } M, \\ w_1, w_2 \in H^1(M): \ \int_M w_1\, d\sigma_g = 0 = \int_M w_2\, d\sigma_g, \end{cases} \tag{6.1.7}$$

where $\mu > 0$, $\lambda > 0$, $\theta \in (0, \frac{\pi}{2})$, and h and f are two weight functions of the type described in (6.1.2). For $\lambda = 0$, the system (6.1.7) decouples and its solvability is equivalent to that of (6.1.1). In fact, the two problems are very much related as we see by the following:

Corollary 6.1.2 *Let h satisfy (6.1.2), (6.1.3), and (6.1.4), and let*

$$f \in C^{\gamma}(M) \text{ for } \gamma \in (0,1],\ 0 < a_1 \leq f \leq a_2 \text{ in } M \text{ and } \int_M \log f \, d\sigma_g = 0. \quad (6.1.8)$$

For a given $\theta \in (0, \frac{\pi}{2})$, $\lambda_0 > 0$ and a compact set $\Lambda \subset \mathbb{R}^+ \setminus \Gamma$ (with Γ as given in (5.7.25)) *there exists a constant $C > 0$ such that every (w_1, w_2) solution of* (6.1.7) *with $\mu \in \Lambda$ and $\lambda \in [0, \lambda_0]$ satisfies*

$$||w_1||_{C^{2,\gamma}(M)} + ||w_2||_{C^{2,\gamma}(M)} \leq C,$$

where C depends only on b_1, b_2, A (in (6.1.4)), $\frac{a_2}{a_1}$, λ_0, and $\mu_0 = \max\{\mu : \mu \in \Lambda\}$.

Proof. We only need to treat the case $\lambda > 0$. By a direct application of the maximum principle to the second equation of (6.1.7), we see that

$$\max_M \left(w_2 - \log \int_M f e^{w_2} \, d\sigma_g \right) \leq \log \left(\frac{1}{|M| a_1} \left(\frac{\mu \cos^2 \theta}{\lambda} + 1 \right) \right). \quad (6.1.9)$$

Therefore, setting

$$g_\lambda = \lambda \left(\frac{f e^{w_2}}{\int_M f e^{w_2} \, d\sigma_g} - \frac{1}{|M|} \right) \in C^{\gamma}(M),\ \gamma \in (0,1],$$

for $\mu \in \Lambda$ and $\lambda \in [0, \lambda_0)$, we see that

$$\int_M g_\lambda \, d\sigma_g = 0 \text{ and } ||g_\lambda||_{L^\infty(M)} \leq c_0,$$

with c_0 depending only on $\frac{a_2}{a_1}$, λ_0, and μ_0. Therefore, if $w_{0,\lambda}$ denotes the unique solution for the problem

$$\begin{cases} -\Delta_g w_{0,\lambda} = g_\lambda \text{ in } M, \\ \int_M w_{0,\lambda} \, d\sigma_g = 0, \end{cases}$$

then

$$||w_{0,\lambda}||_{C^1(M)} \leq C_0$$

$\forall \lambda \in (0, \lambda_0]$, and $C_0 > 0$ depends (as c_0) only on $\frac{a_2}{a_1}$, λ_0, and μ_0. Set $w = w_1 - w_{0,\lambda}$, which satisfies

$$\begin{cases} -\Delta w = \mu \left(\frac{\hat{h} e^{w}}{\int_M \hat{h} e^{w} \, d\sigma_g} - \frac{1}{|M|} \right) \text{ in } M, \\ w \in H^1(M) : \int_M w \, d\sigma_g = 0, \end{cases}$$

with $\hat{h} = he^{w_{0,\lambda}}$. Since $\hat{h}$ keeps the same properties of h, we can apply Proposition 6.1.1 to w and find

$$||w||_{C^{2,\gamma}(M)} \le C_1, \ \gamma \in (0,1),$$

with $C_1 > 0$ depending on $b_1, b_2, A, \frac{a_2}{a_1}, \lambda_0$ and μ_0.

As a consequence, we see that w_1 is uniformly bounded in $C^1(M)$-norm. Substituting this information into problem (6.1.7) together with (6.1.9), we arrive the desired estimate by a bootstrap argument. □

Thus as above, for $\mu \in \mathbb{R}^+ \setminus \Gamma$, the Leray–Schauder degree of the Fredholm map associated to (6.1.7) is well defined at zero.

More precisely, define

$$F_{\mu,\lambda} : E \times E \to E \times E :$$

$$F_{\mu,\lambda}(w_1, w_2) = \left(w_1 + \mu T_h(w_1) + \lambda T_f(w_2), w_2 + \frac{\mu}{2} T_h(w_1) + \frac{\lambda}{2\cos^2\theta} T_f(w_2) \right)$$

where T_h is the compact operator defined in (2.5.6). By virtue of Corollary 6.1.2, for $\mu \in \mathbb{R}^2 \setminus \Gamma$ and $\lambda \in [0, \lambda_0]$ there exists $R > 0$ such that, the Leray–Schauder degree of the map $F_{\mu,\lambda}$ in $\mathcal{B}_R \times \mathcal{B}_R$ is well defined at zero. Here $\mathcal{B}_R$ is the ball around the origin of radius R in the space $E = \{w \in H^1(M) : \int_M w = 0\}$. Furthermore, by the homotopy invariance of the degree, we see that

$$\deg(F_{\mu,\lambda}, \mathcal{B}_R \times \mathcal{B}_R, 0) = \deg(F_{\mu,\lambda=0}, \mathcal{B}_R \times \mathcal{B}_R, 0) = \deg(Id + \mu T_h, \mathcal{B}_R, 0) = d_\mu.$$

In other words, the Leray–Schauder degree of the Fredholm map associated to $F_{\mu,\lambda}$ is independent of λ and coincides with the degree of the Fredholm map relative to (6.1.1). In particular, under the assumptions of Corollary (5.7.11), for every $\lambda \ge 0$ we have

$$\deg(F_{\mu,\lambda}, \mathcal{B}_R \times \mathcal{B}_R, 0) = \begin{cases} 1, & \text{for } \mu \in (0, 8\pi) \\ \chi(M) + m + 1, & \text{for } \mu \in (8\pi, 16\pi) \end{cases} \tag{6.1.10}$$

where, for $\mu \in (8\pi, 16\pi)$, we also suppose that h satisfies (6.1.2) with $\alpha_j \ge 1 \ \forall j = 1 \dots, m$. Clearly, the formula (6.1.10) reduces to (2.5.10) for the case in which h *never* vanishes in M (i.e., when we have $\alpha_j = 0 \ \forall j = 1, \dots, m$ in (6.1.2) and consequently $\Gamma = 8\pi\mathbb{N}$).

As an immediate consequence of (6.1.10), we find

Corollary 6.1.3 *Under the assumptions of Corollary 6.1.2, problem (6.1.7) always admits a solution for* $\mu \in (0, 8\pi)$. *Furthermore, if (6.1.2) holds with* $\alpha_j \ge 1$ *and* $\mu \in (8\pi, 16\pi)$, *then problem (6.1.7) admits a solution provided that M has a positive genus g or that* $M = S^2$ *and* $m \ge 2$.

6.2 An existence result

In this section, we take advantage of the concentration/compactness result of Theorem 5.7.65 to give a variational construction of the solution for problem (6.1.7), as claimed in Corollary 6.1.3 when M admits a positive genus. This construction was available before the degree formula in (6.1.10) had been established in [ChLW]. It is based on an approach used by Ding–Jost–Li–Wang [DJLW3] to handle the Mean Field Equation (6.1.1). In the same spirit, we mention the work of Struwe–Tarantello in [ST] in the periodic setting, concerning the problem:

$$\begin{cases} -\Delta w = \mu \left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{\Omega} \right) \text{ in } \Omega = \left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right], \\ w \text{ doubly periodic in } \partial\Omega, \int_\Omega w = 0, \end{cases} \tag{6.2.1}$$

$0 < a < b$.

In [ST] it was shown that problem (6.2.1) always admits a *non-trivial* solution for $\mu \in \left(8\pi, 4\pi^2 \frac{b}{a}\right)$. Notice that in addition to the trivial solution $w = 0$, problem (6.2.1) also admits one-dimensional solutions (periodically) depending on only *one* of the two variables. The interesting feature about the existence result in [ST], is that it provides a truly 2-dimensional solution for (6.2.1), since non-trivial one-dimensional solutions only exist for $\mu \geq 4\pi^2 \frac{b}{a}$ (see [RT2]).

The goal of this section is to prove the following result:

Theorem 6.2.4 *Let h satisfy (6.1.2) and (6.1.4) and let f satisfy (6.1.8).*

If M has a positive genus $g > 0$ and if $\mu \in (8\pi, 16\pi) \setminus \{8\pi(1+\alpha_j), j = 1, \ldots, m\}$ then problem (6.1.7) admits a solution for every $\lambda \geq 0$ and $\theta \in \left(0, \frac{\pi}{2}\right)$.

We observe that problem (6.1.7) admits a variational formulation in the product space $E \times E$, where $E = \{w \in H^1(M) : \int_M w = 0\}$ defines a Hilbert space equipped with the usual scalar product and norm. For every $(w_1, w_2) \in E \times E$, we define the functional:

$$\begin{aligned} I_\mu(w_1, w_2) = {} & \frac{\tan^2 \theta}{2} \left(\frac{1}{2} \int_M |\nabla_g w_1|^2 \, d\sigma_g - \mu \log \left(\fint_M h e^{w_1} \, d\sigma_g \right) \right) \\ & + \int_M |\nabla_g \left(\frac{w_1}{2} + w_2 \right)|^2 \, d\sigma_g + \lambda \tan^2\theta \log \left(\fint_M f e^{w_2} \, d\sigma_g \right). \end{aligned} \tag{6.2.2}$$

It is easy to check that $I_\mu \in C^1 \, (E \times E)$ and that any critical point $(w_1, w_2) \in E \times E$ for I_μ satisfies:

$$\begin{aligned} \frac{\partial I_\mu}{\partial w_1} (w_1, w_2) \, \varphi = {} & \frac{\tan^2\theta}{2} \int_M \left(\nabla_g w_1 \cdot \nabla_g \varphi - \mu \frac{h e^{w_1}}{\int\limits_M h e^{w_1} \, d\sigma_g} \varphi \right) d\sigma_g \\ & + \int_M \nabla_g \left(\frac{w_1}{2} + w_2 \right) \cdot \nabla_g \varphi \, d\sigma_g = 0, \\ \frac{\partial I_\mu}{\partial w_2} (w_1, w_2) \, \varphi = {} & 2 \int_M \nabla_g \left(\frac{w_1}{2} + w_2 \right) \cdot \nabla_g \varphi \, d\sigma_g \\ & + \lambda \tan^2\theta \int_M \frac{f e^{w_2} \varphi}{\int_M f e^{w_2} \, d\sigma_g} \, d\sigma_g = 0, \end{aligned} \tag{6.2.3}$$

for every $\varphi \in E$.

In other words, critical points of I_μ define (weak) solutions for the 2×2 system of equations:

$$\begin{cases} -\Delta_g \left(\frac{1}{2\cos^2\theta} w_1 + w_2 \right) = \frac{\mu}{2} \tan^2\theta \left(\frac{h e^{w_1}}{\int\limits_M h e^{w_1} \, d\sigma_g} - \frac{1}{|M|} \right) & \text{on } M, \\ \Delta_g \left(\frac{1}{2} w_1 + w_2 \right) = \frac{\lambda}{2} \tan^2\theta \left(\frac{f e^{w_2}}{\int\limits_M f e^{w_2} \, d\sigma_g} - \frac{1}{|M|} \right) & \text{on } M, \\ \int_M w_1 \, d\sigma_g = 0 = \int_M w_2 \, d\sigma_g, \end{cases} \tag{6.2.4}$$

and so they satisfy (6.1.7). Therefore, once we obtain a critical point of I_μ in $E \times E$ we find a solution to (6.1.7).

For this purpose, note that for a given $w_1 \in E$, there exists a *unique* $w_2 \in E$, (depending on w_1) which satisfies (weakly) the second equation in (6.2.4). A simple application of the Implicit Function Theorem also shows that the dependence of w_2 on w_1 is of class C^1. More precisely,

Lemma 6.2.5 *There exists a C^1-map $\gamma : E \to E$, such that*

$$\frac{\partial I_\mu}{\partial w_2} (w, z) = 0 \quad \textit{in} \;\; E^* \;\; \textit{if and only if} \;\; z = \gamma\,(w)\,. \tag{6.2.5}$$

(Recall that E^* denotes the dual space of E.)

Proof: We start with the following:
Claim 1: For every $w_1 \in E$ there exists a unique *$w_2 \in E$ that satisfies:*

$$\frac{\partial I_\mu}{\partial w_2} (w_1, w_2) = 0 \quad \text{in } E^*. \tag{6.2.6}$$

To obtain Claim 1, we fix $w_1 \in E$ and observe that the functional $I_0\,(w) := I_\mu\,(w_1, w)$ is coercive, weakly lower semicontinuous, and bounded below in E. The corresponding minimizer $w_2 \in E$ satisfies (6.2.6). We see that there are no other critical points for I_0 in E. In fact, let $z \in E$ be another critical point for I_0 in E. Therefore it satisfies:

$$\frac{\partial I_\mu}{\partial w_2} (w_1, z) = 0 \text{ in } E^*.$$

Set $\psi = z - w_2$, and consider the function $g \in C^2\,(\mathbb{R}, \mathbb{R})$ defined as follows:

$$g\,(t) = I_\mu\,(w_1, w_2 + t\psi)\,, \;\; t \in \mathbb{R}.$$

We have

$$\dot{g}(0) = \dot{g}\,(1) = 0,$$

and

$$\ddot{g}\,(t) = 2 \int_M |\nabla_g \psi|^2 \, d\sigma_g$$
$$+ \lambda \tan^2\theta \left(\int_M \frac{f e^{w_2 + t\psi}}{\int_M f e^{w_2 + t\psi} \, d\sigma_g} \psi^2 \, d\sigma_g - \left(\int_M \frac{f e^{w_2 + t\psi}}{\int_M f e^{w_2 + t\psi} \, d\sigma_g} \psi \, d\sigma_g \right)^2 \right),$$

$\forall t \in \mathbb{R}$. Notice that, by Jensens's inequality (2.5.17), we have:

$$\int_M \frac{fe^w}{\int_M fe^w\, d\sigma_g}\psi^2\, d\sigma_g - \left(\int_M \frac{fe^w}{\int_M fe^w\, d\sigma_g}\psi\, d\sigma_g\right)^2 \geq 0, \quad \forall\, w, \psi \in E; \quad (6.2.7)$$

and so $\ddot{g}(t) \geq 0$ in $\mathbb{R}$. Consequently $\dot{g} = 0$ is identically zero in [0, 1], and we conclude that necessarily $\psi = 0$. Thus $z = w_2$ as claimed.

Therefore, at every $w_1 \in E$ that the map $\gamma : E \to E$ associates a *unique* w_2 satisfying (6.2.6) in a well defined manner. In other words, (6.2.5) holds.

To show that $\gamma \in C^1(E)$ we apply the Implicit Function theorem (see [Nir]) to the function $F : E \times E \to E^*$,

$$F(w_1, w_2) = \frac{\partial I_\mu}{\partial w_2}(w_1, w_2).$$

Claim 2: $F \in C^1(E \times E, E^*)$ and for every $(w_1, w_2) \in E \times E$ the map: $\frac{\partial F}{\partial w_2}(w_1, w_2)$ defines an isomorphism *from E* onto E^*.

From (6.2.3) it is straightforward to check that F is Frechét differentiable, and for every $(w_1, w_2), (\psi_1, \psi_2) \in E \times E$, we have

$$F'(w_1, w_2)(\psi_1, \psi_2) = \frac{\partial F}{\partial w_1}(w_1, w_2)\, \psi_1 + \frac{\partial F}{\partial w_2}(w_1, w_2)\, \psi_2 \ \in E^*,$$

with

$$\left(\frac{\partial F}{\partial w_1}(w_1, w_2)\, \psi_1\right)\phi = \int_M \nabla_g \psi_1 \cdot \nabla_g \phi\ d\sigma_g,$$

$$\left(\frac{\partial F}{\partial w_2}(w_1, w_2)\, \psi_2\right)\phi = 2\int_M \nabla_g \psi_2 \cdot \nabla_g \phi\ d\sigma_g,$$

$$+\lambda \tan^2\theta \int_M \frac{fe^{w_2}}{\int_M fe^{w_2}\, d\sigma_g}\phi\left(\psi_2 - \frac{fe^{w_2}}{\int_M fe^{w_2}\, d\sigma_g}\psi_2\, d\sigma_g\right) d\sigma_g,$$

for every $\phi \in E$.

Since $\forall\ p \geq 1$, the map : $w \to e^w$ is continuous from E into L^p, we immediately conclude that $F \in C^1(E \times E, E^*)$. Moreover, if we identify in the canonical way E with its dual space E^*, then $\frac{\partial F}{\partial w_2}(w_1, w_2)$ can be identified with a continuous linear operator $A \in B(E, E)$ of the form:

$$A = 2I + K, \quad (6.2.8)$$

where K (depending on w_2 only) is a compact linear map on E.

It remains to be shown that A defines an isomorphism onto E. By the Fredholm alternative this is ensured as soon as we check that $Ker A = 0$. For this purpose, notice that for $\psi \in Ker A$, we have:

$$0 = < A\psi, \psi > = \left(\frac{\partial F}{\partial w_2}(w_1, w_2)\, \psi\right)\psi = 2\int_M |\nabla_g \psi|^2\, d\sigma_g$$

$$+\lambda \tan^2\theta\left(\int_M \frac{fe^{w_2}}{\int_M fe^{w_2}\, d\sigma_g}\psi^2\, d\sigma_g - \left(\int_M \frac{fe^{w_2}}{\int_M fe^{w_2}\, d\sigma_g}\psi\, d\sigma_g\right)^2\right).$$

So by (6.2.7) we immediately obtain $\psi = 0$.

At this point, the conclusion that $\gamma \in C^1(E)$ easily follows from Claim 1 and the Implicit Function theorem applied around each pair $(w_1, \gamma(w_1))$.

By the second equation in (6.2.3), we see that $\gamma(w)$ is *independent* of μ.

For $w \in E$, define the restricted functional:

$$J_\mu(w) = \frac{\tan^2\theta}{2}\left(\frac{1}{2}\int_M |\nabla_g w|^2 d\sigma_g - \mu \ln\left(\fint_M he^w\, d\sigma_g\right)\right)$$
$$+\int_M |\nabla_g\left(\frac{w}{2}+\gamma(w)\right)|^2\, d\sigma_g + \lambda \tan^2\theta \log\left(\fint_M fe^{\gamma(w)}\, d\sigma_g\right).$$

Clearly $J_\mu \in C^1(E)$, and in view of (6.2.5), we have that w defines a critical point for J_μ in E if and only if the pair $(w, \gamma(w))$ is a critical point for I_μ in $E \times E$.

Remark 6.2.6 Notice that if $\mu \in (0, 8\pi)$, then by the Moser–Trudinger inequality (2.4.17), J_μ is coercive and bounded from below in E. A critical point is easily obtained in this case by minimization.

On the contrary, for $\mu > 8\pi$, a critical point for J_μ can only be obtained by introducing a more sophisticated "min-max" construction that relies upon the topological information that M admits a positive genus. To this end, notice first that J_μ is monotone decreasing with respect to μ; since for $w \in E$, by Jensen's inequality, we have

$$\fint_M he^w\, d\sigma_g \geq 1.$$

Also notice that

$$\fint_M fe^{\gamma(w)}\, d\sigma_g \geq 1, \forall w \in E. \tag{6.2.9}$$

Denote by

$$\mathcal{Z} = \{p_1, \dots, p_m\} \subset M$$

the finite set of distinct zeroes of h. Let $X : M \to \mathbb{R}^l$ be the embedding map of M into $\mathbb{R}^l$ $l \geq 3$, and let $\Gamma_1 \subset M \setminus \mathcal{Z}$ be a regular simple closed curve such that its image $\tilde{\Gamma}_1 = X(\Gamma_1)$ *links* with a closed curve $\Gamma_2 \subset \mathbb{R}^l \setminus X(M)$. The existence of Γ_1 and Γ_2 is ensured by the property that M admits a positive genus. For $w \in E$, set

$$m(w) = \frac{\int_M Xe^w\, d\sigma_g}{\int_M e^w\, d\sigma_g} \in \mathbb{R}^l, \tag{6.2.10}$$

the *center of mass* of w.

Denote by $\mathcal{D}_\mu$ the set of continuous maps $g : B_1 \to E$ such that, for $r = |z|$, we have:

(i)

$$\lim_{r\to 1^-} J_\mu(g(z)) = -\infty$$

(ii) the map:

$$m(e^{i\tau}) = \lim_{r\to 1^-} m(g(re^{i\tau}))$$

defines a continuous map from S^1 into $\tilde{\Gamma}_1$ with non-zero degree.

Claim 1. If $\mu > 8\pi$*, then* $\mathcal{D}_\mu$ *is* not *empty.* To establish Claim 1 for any $p \in M \setminus \mathcal{Z}$, we introduce the function $u_{\varepsilon,p}$ defined in a small neighborhood of p, in terms of the isothermal coordinate system $y = (y_1, y_2)$ centered at p, by the expression

$$u_{\varepsilon,p}(y) = \log\left(\frac{\varepsilon}{\left(\varepsilon + \sigma\pi\,|y|^2\right)^2}\right),$$

where $\sigma = h\,(p) > 0$. Denote by $w_{\varepsilon,p} \in E$ the *unique* solution for the problem:

$$\begin{cases} -\Delta_g w_{\varepsilon,p} = 8\pi\left(\frac{\mathcal{X}e^{u_{\varepsilon,p}}}{\int_M \mathcal{X}e^{u_{\varepsilon,p}}\,d\sigma_g} - \frac{1}{|M|}\right) \text{ on } M, \\ \int_M w_{\varepsilon,p}\,d\sigma_g = 0, \end{cases} \tag{6.2.11}$$

where $\mathcal{X}$ denotes a standard cut-off function supported in a small neighborhood of p in M where $u_{\varepsilon,p}$ is defined. Clearly, $w_{\varepsilon,p} \in E$ depends continuously on $\varepsilon > 0$ and $p \in M$. Moreover, by means of Green's representation formula, it is not difficult to show that for $p \in M \setminus \mathcal{Z}$ and $\varepsilon \to 0$, we have:

$$\begin{aligned} &||\nabla_g w_{\varepsilon,p}||^2_{L^2(M)} = 32\pi\,\log\frac{1}{\varepsilon} + O\,(1)\,, \\ &\log\left(⨍_M h e^{w_{\varepsilon,p}}\,d\sigma_g\right) = 2\log\frac{1}{\varepsilon} + O\,(1)\,, \\ &\frac{e^{w_{\varepsilon,p}}}{\int_M e^{w_{\varepsilon,p}}\,d\sigma_g} \to \delta_p, \quad \text{weakly in the sense of measure.} \end{aligned}$$

See [NT2] and [DJLW1].

We notice also that

$$||\nabla_g \frac{w_{\varepsilon,p}}{2} + \gamma\,\left(w_{\varepsilon,p}\right)||^2_{L^2(M)} + \lambda\tan^2\theta\,\log\left(⨍_M e^{\gamma\,\left(w_{\varepsilon,p}\right)}\,d\sigma_g\right) \le C, \tag{6.2.12}$$

for a suitable constant $C > 0$ independent of $\varepsilon > 0$ and $p \in M$.

Indeed, in view of (6.2.11), we find that $\eta = \gamma\,\left(w_{\varepsilon,p}\right)$ satisfies

$$\begin{cases} \Delta_g \eta = 4\pi\,\frac{\mathcal{X}e^{u_{\varepsilon,p}}}{\int_M \mathcal{X}e^{u_{\varepsilon,p}}\,d\sigma_g} + \lambda\,\frac{\tan^2\theta}{2}\,\frac{fe^\eta}{\int_M fe^\eta\,d\sigma_g} - \frac{1}{|M|}\left(4\pi + \frac{\lambda}{2}\tan^2\theta\right) \text{ in } M, \\ \int_M \eta\,d\sigma_g = 0. \end{cases} \tag{6.2.13}$$

Therefore, we find a constant $A > 0$ (independent of ε and p) such that

$$\max_M \eta \le A. \tag{6.2.14}$$

This estimate (together with (6.2.9)) implies that

$$\frac{fe^\eta}{\int_M fe^\eta\,d\sigma_g} \le a_2\frac{e^A}{|M|} = C_1.$$

Since

$$\Delta_g \left(\frac{w_{\varepsilon,p}}{2} + \eta \right) = \frac{\lambda}{2} \tan^2\theta \left(\frac{f e^{\eta}}{\int_M f e^{\eta} \, d\sigma_g} - \frac{1}{|M|} \right),$$

we immediately derive that $\|\nabla_g \left(\frac{w_{\varepsilon,p}}{2} + \eta \right)\|_{L^2(M)} \le C_2$ for a suitable $C_2 > 0$, independent of ε and p, and (6.2.12) follows.

Let us denote by $p = p(\tau)$ $\tau \in [0, 2\pi]$ a regular simple parametrization of Γ_1. In view of (6.2.14) and (6.2.12), we can then easily check that, for $\mu > 8\pi$, the function $h\left(re^{i\tau}\right) := w_{1-r,p(\tau)}$ $r \in [0,1]$ and $\tau \in [0, 2\pi)$ belongs to $\mathcal{D}_\mu$. Claim 1 is thus established.

At this point, we may follow [DJLW3] and define

$$c_\mu = \inf_{g \in \mathcal{D}_\mu} \sup_{w \in g(B_1)} J_\mu(w) \tag{6.2.15}$$

as a "good" candidate for a critical value of J_μ.

Claim 2. If $\mu \in (8\pi, 16\pi)$, then $c_\mu > -\infty$. The proof of Claim 2 relies in an essential way, upon the following improved form of Moser–Trudinger's inequality (see e.g., [CL1] and [DM] for generalizations).

Lemma 6.2.7 *Let S_1 and S_2 be two subsets of M satisfying dist$(S_1, S_2) \ge \delta_0 > 0$, and let $\gamma_0 \in \left(0, \frac{1}{2}\right)$. For any $\varepsilon > 0$, there exists a constant $c = c(\varepsilon, \delta_0, \gamma_0) > 0$ such that, for all $u \in E$ satisfying*

$$\frac{\int_{S_1} e^u \, d\sigma_g}{\int_M e^u \, d\sigma_g} \ge \gamma_0, \qquad \frac{\int_{S_2} e^u \, d\sigma_g}{\int_M e^u \, d\sigma_g} \ge \gamma_0, \tag{6.2.16}$$

we have,

$$\int_M e^u \, d\sigma_g \le c \exp\left(\frac{1}{32\pi - \varepsilon} \int_M |\nabla_g u|^2 \, d\sigma_g \right). \tag{6.2.17}$$

A simple application of Holder's inequality shows that, for $\mu < 16\pi$, J_μ is bounded below and coercive on the set of functions satisfying (6.2.16).

Notice also that, for any $g \in \mathcal{D}_\mu$ there exists a function $w \in g(B_1)$ with $m(w) \in \Gamma_2$. So, if by contradiction, we assume that (6.2.15) yields to the value $-\infty$, then we would find a sequence $w_n \in E$, with $m(w_n) \in \Gamma_2$, and

$$J_\mu(w_n) \to -\infty. \tag{6.2.18}$$

Hence, w_n must violate (6.2.16). So there must exist a point $p_0 \in M$, such that for every $\delta > 0$, we have

$$\frac{\int_{U_\delta(p_0)} e^{w_n} \, d\sigma_g}{\int_M e^{w_n} \, d\sigma_g} \to 1, \quad \text{as } n \to \infty,$$

with $U_\delta(p_0) = \{p \in M : d_g(p, p_0) < \delta\}$. Consequently,

$$|m(w_n) - X(p_0)| \leq \int_{U_\varepsilon(p_0)} |X - X(p_0)| \frac{e^{w_n}\, d\sigma_g}{\int_M e^{w_n}\, d\sigma_g} + o(1)$$
$$\leq \sup_{p \in U_\delta(p_0)} |X(p) - X(p_0)| + o(1), \text{ as } n \to \infty.$$

But this is clearly impossible, since by choosing $\delta > 0$ sufficiently small, the estimate above implies that $|m(w_n) - X(p_0)| < \frac{1}{2}\operatorname{dist}(\Gamma_2, X(M))$ for n large, in contradiction with the fact that $m(w_n) \in \Gamma_2$ and $p_0 \in M$.

Claim 3. There exists a dense *set* $\Lambda \subset (8\pi, 16\pi)$ *such that for every* $\mu \in \Lambda$, c_μ *defines a critical value for* J_μ *in* E.

To establish Claim 3, as in [ST] and [DJLW3], we shall use Struwe's monotonicity trick. (See [St2], [St3], and [Je] for a clean presentation of this technique in a useful general framework.) For this purpose, note that if $\mu_1 < \mu_2$, then $\mathcal{D}_{\mu_1} \subset \mathcal{D}_{\mu_2}$ and $c_{\mu_1} \geq c_{\mu_2}$. Hence c_μ defines a non-increasing function of μ.

Denote by $\Lambda \subset (8\pi, 16\pi)$ the set of the values μ, where c_μ is differentiable. We know that Λ is dense in $[8\pi, 16\pi]$.

Thus for any $\mu \in \Lambda$, we can choose a sequence $\mu_n \nearrow \mu$ such that

$$0 \leq \frac{c_{\mu_n} - c_\mu}{\mu - \mu_n} \leq C, \tag{6.2.19}$$

for some constant C independent of n.

Furthermore, for $g_n \in \mathcal{D}_{\mu_n} \subset \mathcal{D}_\mu$ satisfying

$$\max_{w \in g_n(\overline{B_1})} J_{\mu_n}(w) \leq c_{\mu_n} + \mu - \mu_n, \tag{6.2.20}$$

and for all $w \in g_n(\overline{B_1})$ such that

$$J_\mu(w) \geq c_\mu - (\mu - \mu_n), \tag{6.2.21}$$

we have:

$$\log\left(\fint_M h e^w\, d\sigma_g\right) = 2\text{cotan}^2\theta \left(\frac{J_{\mu_n}(w) - J_\mu(w)}{\mu - \mu_n}\right) \leq 2\text{cotan}^2\theta \left(\frac{c_{\mu_n} - c_\mu}{\mu - \mu_n}\right)$$
$$\leq 2\text{cotan}^2\theta\,(C + 1) \equiv C_1.$$

Consequently,

$$\begin{aligned} ||\nabla_g w||^2_{L^2(M)} &\leq \left(4\text{cotan}^2\theta\right) J_{\mu_n}(w) + 2\mu_n \log\left(\fint_M h e^w\, d\sigma_g\right) \\ &\leq 4\text{cotan}^2\theta\left(c_{\mu_n} + \mu - \mu_n\right) + 2\mu_n C_1. \end{aligned} \tag{6.2.22}$$

In view of (6.2.19), we have

$$c_{\mu_n} \leq c_\mu + C(\mu - \mu_n), \tag{6.2.23}$$

and from (6.2.22) we find a constant $R > 0$, independent of n, such that if (6.2.20) holds, then for every $w \in g_n(\overline{B_1})$ satisfying (6.2.20), we have $||\nabla_g w||_{L^2(M)} \leq R$.

We are going to show that there exists a Palais–Smale sequence $\{w_n\} \subset E$ for J_μ at level c_μ (see (2.3.2)) with $||\nabla_g w_n||_{L^2(M)} \leq R$. For this purpose, we argue by contradiction, and suppose there exists $\delta > 0$ such that every $w \in E : ||\nabla_g w||_{L^2(M)} \leq R$ and $|J_\mu(w) - c_\mu| < \delta$ also satisfies $||J_\mu'(w)||_{E^*} \geq \delta$.

By means of a pseudo-gradient flow (see Theorem 2.3.4 in Chapter 2), for every $\overline{\varepsilon} \in (0, \delta)$, there exists $\varepsilon \in (0, \overline{\varepsilon})$ and an homeomorphism $\eta : E \to E$ with the following properties:

(i) $\eta(w) = w$, provided $|J_\mu(w) - c_\mu| \geq \overline{\varepsilon}$;
(ii) $J_\mu(\eta(w)) \leq J_\mu(w)$;
(iii) if $||\nabla_g w||_{L^2(M)} \leq R$ and $J_\mu(w) \leq c_\mu + \varepsilon$, then $J_\mu(\eta(w)) \leq c_\mu - \varepsilon$.

Let us choose $g_n \in \mathcal{D}_{\mu_n} \subset \mathcal{D}_\mu$ so that (6.2.20) holds. In view of properties (i) and (ii) above, we also have that $\eta \circ g_n \in \mathcal{D}_{\mu_n} \subset \mathcal{D}_\mu$ satisfies (6.2.20). Let $w_n \in g_n(\overline{B_1})$ be such that $J_\mu(w_n) = \max_{w \in g_n(\overline{B_1})} J_\mu(w)$. By definition $J_\mu(w_n) \geq c_\mu$, so (6.2.20) and (6.2.21) hold for w_n and imply that $||\nabla_g w_n||_{L^2(M)} \leq R$. Furthermore, by (6.2.20) and (6.2.23), for n large we have:

$$\max_{w \in g_n(\overline{B_1})} J_\mu(w) = J_\mu(w_n) \leq J_{\mu_n}(w_n) \leq c_{\mu_n} + \mu - \mu_n \leq c_\mu + (C+1)(\mu - \mu_n) < c_\mu + \varepsilon.$$

Hence, for n large, we can use (iii) for w_n to see that

$$J_\mu(\eta(w_n)) \leq c_\mu - \varepsilon.$$

Consequently,

$$\max_{w \in \eta \circ g_n(\overline{B_1})} J_\mu(w) \leq c_\mu - \varepsilon$$

in contradiction to the definition of c_μ.

In conclusion, for every $\mu \in \Lambda$, there exists a sequence $w_n \in E$ that satisfies:

$$||\nabla_g w_n||_{L^2(M)} \leq C_2, \ J_\mu(w_n) \to c_\mu \text{ and } J_\mu'(w_n) \to 0 \text{ in } E^*. \tag{6.2.24}$$

We show how properties (6.2.24) imply that $\{w_n\}$ admits a strongly convergent subsequence to a critical point of J_μ with a corresponding critical value c_μ.

Indeed, if $\{w_n\}$ satisfies (6.2.24), then both w_n and $z_n := \gamma(w_n)$ define uniformly bounded sequences in E. Hence, passing to a subsequence (denoted in the same way), we find $(w, z) \in E$ such that:

$$\begin{aligned} &(w_n, z_n) \to (w, z), \text{ weakly in } E \times E \text{ and strongly in } L^p(M) \times L^p(M), \ \forall p \geq 1, \\ &(e^{w_n}, e^{z_n}) \to (e^w, e^z), \text{ strongly in } L^p(M) \times L^p(M), \ \forall p \geq 1. \end{aligned} \tag{6.2.25}$$

Furthermore, by passing to the limit into the relation

$$\langle \frac{\partial I_\mu}{\partial w_2}(w_n, z_n), \varphi \rangle = 0, \ \forall \varphi \in E,$$

we find that $\frac{\partial I_\mu}{\partial w_2}(w, z) = 0$. Therefore, by Lemma 6.2.5 we deduce that $z = \gamma(w)$. Note also that as $n \to \infty$,

$$2\int_M \nabla_g \left(\frac{w_n}{2} + z_n\right) \cdot \nabla_g (w_n - w)\, d\sigma_g = -\lambda \tan^2\theta \int_M \frac{f e^{z_n}}{\int_M f e^{z_n}\, d\sigma_g} (w_n - w)\, d\sigma_g$$
$$= o(1),$$

and

$$|J_\mu^{'}(w_n)(w_n - w)| \le ||J_\mu^{'}(w_n)||_{E^*} ||\nabla_g (w_n - w)||_{L^2(M)} = o(1).$$

Therefore, as $n \to \infty$,

$$\begin{aligned} o(1) &= |J_\mu^{'}(w_n)(w_n - w)| \\ &= \frac{\tan^2\theta}{2}\left(\int_M \nabla_g w_n \cdot \nabla_g (w_n - w)\, d\sigma_g - \mu \frac{\int_M h e^{w_n}(w_n - w)\, d\sigma_g}{\int_M h e^{w_n}\, d\sigma_g}\right) \\ &\quad + 2\int_M \nabla_g\left(\frac{w_n}{2} + z_n\right) \cdot \nabla_g (w_n - w)\, d\sigma_g + \lambda\tan^2\theta \frac{\int_M f e^{z_n}(w_n - w)\, d\sigma_g}{\int_M f e^{z_n}\, d\sigma_g} \\ &= \frac{\tan^2\theta}{2}\left(||\nabla_g w_n||^2_{L^2(M)} - ||\nabla_g w||^2_{L^2(M)}\right) + o(1). \end{aligned} \tag{6.2.26}$$

Consequently, $w_n \to w$ and $\gamma(w_n) \to \gamma(w)$ strongly in E, $c_\mu = J_\mu(w)$ and $J_\mu^{'}(w) = 0$. □

Proof of Theorem 6.2.4. From Claim 3, we know that problem (6.1.7) admits a solution for every $\mu \in \Lambda$. Next we use the uniform estimates of Corollary 6.1.2 to show that this actually remains true for any $\mu \in (8\pi, 16\pi) \setminus \{8\pi(1 + \alpha_j),\ j = 1, \ldots, m\} \subset \mathbb{R}^+ \setminus \Gamma$. To this purpose, let $\mu_n \in \Lambda$ be such that $\mu_n \to \mu$, and denote by $(w_{1,n}, w_{2,n})$ the solution of (6.1.7) with $\mu = \mu_n$. By Corollary 6.1.2, we know that $(w_{1,n}, w_{2,n})$ defines a uniformly bounded sequence in $C^{2,\gamma}(M) \times C^{2,\gamma}(M)$. Hence, by passing to a subsequence if necessary, we find that $(w_{1,n}, w_{2,n}) \to (w_1, w_2)$ in $C^2(M) \times C^2(M)$, and (w_1, w_2) gives the desired solution. □

6.3 Extremals for the Moser–Trudinger inequality in the periodic setting

In this section, we investigate the existence of periodic extremals for the Moser–Trudinger inequality (2.4.17), where the closed surface M is the flat 2-torus. For simplicity, we shall focus on the situation where

$$M = \mathbb{R}^2 / a\mathbb{Z} \times b\mathbb{Z}\, (\simeq \mathbb{C} / a\mathbb{Z} + ib\mathbb{Z}), \tag{6.3.1}$$

$a > 0$ and $b > 0$, and thus take as a periodic cell domain:

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : |x| \leq \frac{a}{2},\ |y| \leq \frac{b}{2} \right\} \simeq \left\{ z = at + ibs,\ t, s \in \left(-\frac{1}{2}, \frac{1}{2} \right) \right\}. \quad (6.3.2)$$

We recall, that in this case, the Green's function (cf. (2.5.11)) is identified by an Ω-periodic function $G = G(z)$ given as

$$G(z) = \frac{1}{2\pi} \log \frac{1}{|z|} + \gamma(z), \quad (6.3.3)$$

with $\gamma = \gamma(z)$ explicitly defined in (2.5.15).

Let

$$h = e^{u_0} \in C^0(M) \text{ such that } u_0 \in L^1(M) : \int_M u_0 \, d\sigma_g = 0. \quad (6.3.4)$$

Notice in particular that e^{u_0} in (6.3.4) attains its maximum value at a point $p_0 \in M$ where u_0 is continuous.

By the Moser–Trudinger inequality (2.4.17), the functional

$$I(w) = \frac{1}{2} ||\nabla w||^2_{L^2(M)} - 8\pi \log ⨍_M e^{u_0 + w} \, d\sigma_g \quad (6.3.5)$$

is bounded from below in $E = \{w \in H^1(M) : \int_M w = 0\}$.

Thus, investigating the existence of an extremal for the Moser–Trudinger inequality becomes equivalent to obtaining a minimum point for I in E.

In this direction we prove:

Theorem 6.3.8 *Let M be as given in (6.3.1) and let u_0 satisfy (6.3.4). For $p_0 \in M$: $u_0(p_0) = \max_M u_0$, we assume that $u_0 \in C^2(U_\delta(p_0))$ and that*

$$\Delta u_0(p_0) + \frac{8\pi}{|M|} > 0. \quad (6.3.6)$$

Then I attains *its infimum in E.*

Remark 6.3.9 (a) By virtue of a recent result of Chen–Lin–Wang [ChLW], we actually know the condition (6.3.6) is *sharp*. In fact, if we choose as u_0 the *unique* solution for the problem:

$$\begin{cases} \Delta u_0 = 8\pi \delta_{z=0} - \frac{8\pi}{|\Omega|} \text{ in } \Omega, \\ \int_\Omega u_0 = 0, \ u_0 \text{ doubly periodic on } \partial\Omega, \end{cases} \quad (6.3.7)$$

(see 2.5.5), then (6.3.6) is just missed, as we have:

$$\Delta u_0(p_0) + \frac{8\pi}{|\Omega|} = 0, \quad \text{for } p_0 : \ u_0(p_0) = \max_M p_0.$$

It is shown in [ChLW] that in this case the functional I in (6.3.5) *cannot* attain its infimum in E.

More recent contributions towards the existence or non-existence of critical points for the functional I in (6.3.5) and their characterization as minimizers can be found in [LiW] and [LiW1].

(b) Theorem 6.3.8 remains valid for a general surface M, where condition (6.3.6) is replaced by an equivalent condition involving the Gauss curvature of M; see [DJLW1] and [ChL2].

To clarify the role of condition (6.3.6) observe that:

Proposition 6.3.10 *Let M in (6.3.1) and u_0 satisfy (6.3.4). Then*

$$\inf_E I \le -8\pi\left(4\pi\gamma(0) + \max_M u_0 + \log(\frac{\pi}{|M|}) + 1\right) \\ -4e^{-u_0(p_0)}\varepsilon^2 \log\frac{1}{\varepsilon}\left(\Delta u_0(p) + \frac{8\pi}{|M|}\right) + O(\varepsilon^2) \text{ as } \varepsilon \to 0^+, \tag{6.3.8}$$

with γ the regular part of the Green function (given in (2.5.15)).

Proof. To establish (6.3.8), we are going to consider our functions as doubly periodic functions (with the periodic cell domain Ω in (6.3.2)) extended by periodicity over the plane. Moreover, without loss of generality, after a translation and scaling of coordinates, we can assume that $p_0 = 0 \in \Omega$ and $|\Omega| = 1$ (i.e., $b = \frac{1}{a}$ in (6.3.2)). For $\varepsilon > 0$ and $z \in \Omega$, let

$$u_\varepsilon(z) = \log\left(\frac{\varepsilon^2}{(\varepsilon^2 + \pi\sigma|z|^2)^2}\right) \text{ with } \sigma = e^{\max_M u_0}.$$

Define w_ε as the unique solution for the problem:

$$\begin{cases} -\Delta w_\varepsilon = 8\pi\left(\frac{e^{u_0+u_\varepsilon}}{\int_\Omega e^{u_0+u_\varepsilon}} - \frac{1}{|\Omega|}\right) \text{ in } \Omega, \\ w_\varepsilon \text{ doubly periodic on } \partial\Omega, \ \int_\Omega w_\varepsilon = 0 \end{cases} \tag{6.3.9}$$

With a slight abuse of notations, set

$$I(w_\varepsilon) = \frac{1}{2}\int_\Omega |\nabla w_\varepsilon|^2 - 8\pi \log\left(\int_\Omega e^{u_0+u_\varepsilon}\right).$$

Clearly: $\inf_E I \le I(w_\varepsilon)$. We shall establish (6.3.8) by showing that

$$I(w_\varepsilon) \le -8\pi(4\pi\gamma(0) + u_0(0) + \log(\pi) + 1) \\ -\frac{4}{\sigma}\varepsilon^2 \log\frac{1}{\varepsilon}(\Delta u_0(0) + 8\pi) + O(\varepsilon^2) \text{ as } \varepsilon \to 0. \tag{6.3.10}$$

To this purpose, on the set $\Omega_\varepsilon = \{z : \ \varepsilon z \in \Omega\}$ define

$$\rho_\varepsilon = \varepsilon^2 e^{u_0(\varepsilon x)+u_\varepsilon(\varepsilon x)} = \frac{e^{u_0(\varepsilon x)}}{(1+\sigma\pi|x|^2)^2}, \text{ and } A_\varepsilon = \int_\Omega e^{u_0+u_\varepsilon} = \int_{\Omega_\varepsilon} \rho_\varepsilon. \tag{6.3.11}$$

We have

$$
\begin{aligned}
||\nabla w_\epsilon||^2_{L^2} &= -\int_\Omega w_\epsilon \Delta w_\epsilon = \frac{(8\pi)^2}{A_\epsilon^2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} G(\epsilon x, \epsilon y)\rho_\epsilon(x)\rho_\epsilon(y)dxdy\\
&= 32\pi \log \frac{1}{\epsilon} + \frac{32\pi}{A_\epsilon^2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} \log \frac{1}{|x-y|}\rho_\epsilon(x)\rho_\epsilon(y)dxdy\\
&\quad + \frac{(8\pi)^2}{A_\epsilon^2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} \gamma(\epsilon(x-y))\rho_\epsilon(x)\rho_\epsilon(y)dxdy,
\end{aligned}
$$

and

$$
8\pi \log \int_\Omega h e^{w_\epsilon} = -16\pi \log \frac{1}{\epsilon} + 8\pi \log \int_{\Omega_\epsilon} h(\epsilon x)e^{w_\epsilon(\epsilon x)}dx. \tag{6.3.12}
$$

To estimate the second term in (6.3.12), observe that

$$
\begin{aligned}
h(\epsilon x)e^{w_\epsilon(\epsilon x)} &= \rho_\epsilon(x)(1+\sigma\pi|x|^2)^2 e^{w_\epsilon(\epsilon x)}\\
&= \rho_\epsilon(x)(1+\sigma\pi|x|^2)^2\frac{1}{\epsilon^4}\exp\left(\frac{4}{A_\epsilon}\int_{\Omega_\epsilon}\log\left(\frac{1}{|x-y|}\right)\rho_\epsilon(y)dy\right)\\
&\quad \times \exp\left(\frac{8\pi}{A_\epsilon}\int_{\Omega_\epsilon}\gamma(\epsilon(x-y))\rho_\epsilon(y)dy\right).
\end{aligned}
$$

Thus, by Jensen's inequality (2.5.6), we get

$$
\begin{aligned}
&8\pi \log \int_{\Omega_\epsilon} h(\epsilon x)e^{w_\epsilon(\epsilon x)}\\
&= 32\pi \log \frac{1}{\epsilon} + 8\pi \log A_\epsilon + 8\pi \log \left(\frac{1}{A_\epsilon}\int_{\Omega^\epsilon}\rho_\epsilon(x)(1+\sigma\pi|x|^2)^2\right.\\
&\quad \left.\times \exp\left(\frac{4}{A_\epsilon}\int_{\Omega^\epsilon}\ln\frac{1}{|x-y|}\rho_\epsilon(y)dy\right)\exp\left(\frac{8\pi}{A_\epsilon}\int_{\Omega_\epsilon}\gamma(\epsilon(x-y))\rho_\epsilon(y)dy\right)dx\right)\\
&\geq 32\pi \log \frac{1}{\epsilon} + 8\pi \log A_\epsilon + \frac{32\pi}{A_\epsilon^2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_\epsilon(x)\rho_\epsilon(y)dxdy\\
&\quad + \frac{(8\pi)^2}{A_\epsilon^2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\gamma(\epsilon(x-y))\rho_\epsilon(x)\rho_\epsilon(y)dxdy\\
&\quad + \frac{16\pi}{A_\epsilon}\int_{\Omega_\epsilon}\rho_\epsilon(x)\log(1+\pi\sigma|x|^2)dx.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
I(w_\epsilon) \leq &-\frac{16\pi}{A_\epsilon^2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_\epsilon(x)\rho_\epsilon(y)dxdy\\
&-\frac{16\pi}{A_\epsilon}\int_{\Omega_\epsilon}\rho_\epsilon(x)\log(1+\pi\sigma|x|^2)dx\\
&-8\pi \log A_\epsilon - \frac{(8\pi)^2}{2A_\epsilon^2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\gamma(\epsilon(x-y))\rho_\epsilon(x)\rho_\epsilon(y)dxdy.
\end{aligned}
$$

Note that, the first term in the above estimate can be written as

$$\begin{aligned}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_\epsilon(x)\rho_\epsilon(y)dxdy &= \int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_0(x)\rho_0(y)dxdy\\ &\quad -2\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_0(x)\beta_\epsilon(y)dxdy\\ &\quad +\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\log\frac{1}{|x-y|}\beta_\epsilon(x)\beta_\epsilon(y)dxdy,\end{aligned}$$

with

$$\rho_0(x)=\frac{e^{u_0(0)}}{(1+\sigma\pi|x|^2)^2}\qquad\text{and}\qquad\beta_\epsilon(x)=\frac{e^{u_0(0)}-e^{u_0(\epsilon x)}}{(1+\sigma\pi|x|^2)^2}.$$

Since, $\frac{1}{2}\log(\frac{\pi\sigma}{1+\sigma\pi|x|^2})=\int_{\mathbb{R}^2}\log\frac{1}{|x-y|}\rho_0(y)dy$, we derive:

$$\begin{aligned}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_0(x)\rho_0(y)dxdy &= \frac{1}{2}\log(\pi\sigma)\left(\int_{\Omega_\epsilon}\rho_0(x)-\int_{\mathbb{R}^2\setminus\Omega_\epsilon}\rho_0(x)\right)\\ &\quad -\frac{1}{2}\int_{\Omega_\epsilon}\rho_0(x)\log(1+\sigma\pi|x|^2)\\ &\quad +\frac{1}{2}\int_{\mathbb{R}^2\setminus\Omega_\epsilon}\rho_0(x)\log(1+\sigma\pi|x|^2)\\ &\quad +\int_{\mathbb{R}^2\setminus\Omega_\epsilon}\int_{\mathbb{R}^2\setminus\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_0(x)\rho_0(y)dxdy;\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_0(x)\beta_\epsilon(y)dxdy &= \frac{1}{2}\log(\pi\sigma)\int_{\Omega_\epsilon}\beta_\epsilon(y)\\ &\quad -\frac{1}{2}\int_{\Omega_\epsilon}\beta_\epsilon(y)\log(1+\sigma\pi|y|^2)\\ &\quad -\int_{\Omega_\epsilon}\beta_\epsilon(y)(\int_{\mathbb{R}^2\setminus\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_0(x)dx)dy.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}\log\frac{1}{|x-y|}\rho_\epsilon(x)\rho_\epsilon(y)dxdy\\ &\quad=\frac{1}{2}\log(\pi\sigma)\left(\int_{\Omega_\epsilon}\rho_0(x)-2\int_{\Omega_\epsilon}\beta_\epsilon(x)-\int_{\mathbb{R}^2\setminus\Omega_\epsilon}\rho_0(x)\right)\\ &\quad\quad-\frac{1}{2}\int_{\Omega_\epsilon}\rho_0(x)\log(1+\sigma\pi|y|^2)+\int_{\Omega_\epsilon}\beta_\epsilon(x)\log(1+\sigma\pi|x|^2)\\ &\quad\quad+\frac{1}{2}\int_{\mathbb{R}^2\setminus\Omega_\epsilon}\rho_0(x)\log(1+\sigma\pi|x|^2)+\mathcal{R}_\epsilon,\end{aligned}$$

with

$$\mathcal{R}_\epsilon = \int_{\mathbb{R}^2\setminus\Omega_\epsilon}\int_{\mathbb{R}^2\setminus\Omega_\epsilon} \log \frac{1}{|x-y|}\rho_0(x)\rho_0(y)dxdy$$
$$+2\int_{\Omega_\epsilon} \beta_\epsilon(y)(\int_{\mathbb{R}^2\setminus\Omega_\epsilon} \log \frac{1}{|x-y|}\rho_0(x)dx)dy$$
$$+\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} \log \frac{1}{|x-y|}\beta_\epsilon(y)\beta_\epsilon(x)dxdy.$$

We introduce the following notation:

$$\alpha_\epsilon = \int_{\mathbb{R}^2\setminus\Omega_\epsilon} \rho_0(x) \qquad \beta_\epsilon = \int_{\Omega_\epsilon} \beta_\epsilon(x)$$
$$\bar{\alpha}_\epsilon = \int_{\mathbb{R}^2\setminus\Omega_\epsilon} \rho_0(x) \log (1+\sigma\pi|x|^2) \quad \bar{\beta}_\epsilon = \int_{\Omega_\epsilon} \beta_\epsilon(x) \log (1+\sigma\pi|x|^2).$$

Since $\int_{\mathbb{R}^2} \rho_0(x) = 1 = \int_{\mathbb{R}^2} \rho_0(x) \log (1+\sigma\pi|x|^2)$, we may write

$$A_\epsilon \equiv \int_{\Omega_\epsilon} \rho_\epsilon(x) = 1-\alpha_\epsilon-\beta_\epsilon,$$

and

$$\int_{\Omega_\epsilon} \rho_\epsilon(x) \log (1+\sigma\pi|x|^2) = 1-\bar{\alpha}_\epsilon-\bar{\beta}_\epsilon;$$

and obtain

$$I(w_\epsilon) \le -\frac{(8\pi)^2}{2}\gamma(0) - 8\pi \log(1-\alpha_\epsilon-\beta_\epsilon)$$
$$-\frac{8\pi}{(1-\alpha_\epsilon-\beta_\epsilon)^2}\log(\pi\sigma)(1-2\alpha_\epsilon-2\beta_\epsilon)$$
$$+\frac{8\pi}{(1-\alpha_\epsilon-\beta_\epsilon)^2}(1-2\bar{\alpha}_\epsilon-2\bar{\beta}_\epsilon) - \frac{16\pi}{(1-\alpha_\epsilon-\beta_\epsilon)}(1-\bar{\alpha}_\epsilon-\bar{\beta}_\epsilon)$$
$$-\frac{(8\pi)^2}{2(1-\alpha_\epsilon-\beta_\epsilon)^2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}(\gamma(\epsilon(x-y))-\gamma(0))\rho_\epsilon(x)\rho_\epsilon(y) - \frac{16\pi}{(1-\alpha_\epsilon-\beta_\epsilon)^2}\mathcal{R}_\epsilon.$$

That is

$$I(w_\epsilon) \le -8\pi(4\pi\gamma(0)+\log(\pi\sigma)+1)+\Delta_\epsilon, \tag{6.3.13}$$

with

$$\Delta_\epsilon = -8\pi \log(1-\alpha_\epsilon-\beta_\epsilon) + 8\pi\left(\frac{\alpha_\epsilon+\beta_\epsilon}{1-\alpha_\epsilon-\beta_\epsilon}\right)^2 \log \pi\sigma$$
$$+\frac{8\pi}{(1-\alpha_\epsilon-\beta_\epsilon)^2}((\alpha_\epsilon+\beta_\epsilon)(\alpha_\epsilon+\beta_\epsilon-2(\bar{\alpha}_\epsilon+\bar{\beta}_\epsilon))) \tag{6.3.14}$$
$$-\frac{(8\pi)^2}{2(1-\alpha_\epsilon-\beta_\epsilon)^2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon}(\gamma(\epsilon(x-y))-\gamma(0))\rho_\epsilon(x)\rho_\epsilon(y) - \frac{16\pi}{(1-\alpha_\epsilon-\beta_\epsilon)^2}\mathcal{R}_\epsilon,$$

and $\Delta_\epsilon \to 0$ as $\epsilon \to 0$.

In order to find the explicit expression for Δ_ϵ as $\epsilon \to 0$, notice that

$$\begin{aligned}
\alpha_\epsilon &= \int_{\mathbb{R}^2\setminus\Omega_\epsilon} \rho_0(x) = \frac{\epsilon^2}{\sigma\pi^2}\int_{\mathbb{R}^2\setminus\Omega} \frac{1}{|x|^4} + o(\epsilon^2);\\
\beta_\epsilon &= -\frac{\epsilon^2}{2}\left(\partial_1^2 u_0(0)\int_{\Omega_\epsilon} \frac{\sigma|x_1|^2}{(1+\pi\sigma|x|^2)^2} + \partial_2^2 u_0(0)\int_{\Omega_\epsilon} \frac{\sigma|x_2|^2}{(1+\pi\sigma|x|^2)^2}\right) \qquad (6.3.15)\\
&\qquad + O(\epsilon^2)\\
&= O\left(\epsilon^2 \log\frac{1}{\epsilon}\right).
\end{aligned}$$

For a more precise expression for β_ϵ, see (6.3.19) below.

Furthermore,

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus\Omega_\epsilon}\int_{\mathbb{R}^2\setminus\Omega_\epsilon} \log|x-y|\rho_0(x)\rho_0(y) \le 2\alpha_\epsilon\left(\int_{\mathbb{R}^2\setminus\Omega_\epsilon} \log(2|x|)\rho_0(x)\right);\\
&\int_{\mathbb{R}^2\setminus\Omega_\epsilon} \rho_0(x)\int_{\Omega_\epsilon} \log\ |x-y|\beta_\epsilon(y) \le \alpha_\epsilon \int_{\Omega_\epsilon} \log(1+2|y|)\beta_\epsilon(y)\\
&\qquad\qquad + \beta_\epsilon\int_{\mathbb{R}^2\setminus\Omega_\epsilon} \log(2|x|)\rho_0(x);\\
&\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} \log\ |x-y|\beta_\epsilon(x)\beta_\epsilon(y) \le 2\beta_\epsilon\int_{\Omega_\epsilon} \log(1+2|x|)\beta_\epsilon(x).
\end{aligned}$$

And since $\int_{\mathbb{R}^2\setminus\Omega_\epsilon} \log(2|x|)\rho_0(x) = o(\epsilon) = \int_{\Omega_\epsilon} \log(1+2|x|)\beta_\epsilon(x)$, we obtain that $\frac{-\mathcal{R}_\epsilon}{(1-\alpha_\epsilon-\beta_\epsilon)^2} \le o(\epsilon^2)$, as $\epsilon \to 0$.

Consequently, from (6.3.14), (6.3.15) and the fact that $\bar{\alpha}_\epsilon = o(\epsilon) = \bar{\beta}_\epsilon$, it follows that

$$\Delta_\epsilon \le +8\pi(\alpha_\epsilon+\beta_\epsilon) - \frac{(8\pi)^2}{2}\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} (\gamma(\epsilon(x-y)) - \gamma(0))\rho_\epsilon(x)\rho_\epsilon(y) + o(\epsilon^2), \qquad (6.3.16)$$

as $\epsilon \to 0$.

On the other hand,

$$\begin{aligned}
\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} (\gamma(\epsilon(x-y)) - \gamma(0))\rho_\epsilon(x)\rho_\epsilon(y) &= \epsilon^2\partial_1^2\gamma(0)\int_{\Omega_\epsilon} \frac{\sigma x_1^2}{(1+\pi\sigma|x|^2)^2}dx_1dx_2\\
&\quad + \epsilon^2\partial_2^2\gamma(0)\int_{\Omega_\epsilon} \frac{\sigma x_2^2}{(1+\pi\sigma|x|^2)^2}dx_1dx_2\\
&\quad + O(\epsilon^2), \qquad (6.3.17)
\end{aligned}$$

as $\epsilon \to 0$.

Putting together (6.3.15), (6.3.16), and (6.3.17), we conclude,

$$\begin{aligned}\Delta_\epsilon \le -4\pi\epsilon^2 \Bigg[&(8\pi\partial_1^2\gamma(0) + \partial_1^2 u_0(0)) \int_{\Omega_\epsilon} \frac{\sigma x_1^2}{(1+\pi\sigma|x|^2)^2} \\ &+ (8\pi\partial_2^2\gamma(0) + \partial_2^2 u_0(0)) \int_{\Omega_\epsilon} \frac{\sigma x_2^2}{(1+\pi\sigma|x|^2)^2} \Bigg] + O(\epsilon^2),\end{aligned} \tag{6.3.18}$$

as $\epsilon \to 0$.

It is not difficult to check that

$$\begin{aligned}\int_{\Omega_\epsilon} \frac{\sigma x_1^2}{(1+\pi\sigma|x|^2)^2} &= \frac{4}{\sigma\pi^2}\left(\frac{\pi}{4}\log\frac{\sqrt{\pi\sigma}}{\epsilon} + \Gamma_a\right) \\ \int_{\Omega_\epsilon} \frac{\sigma x_2^2}{(1+\pi\sigma|x|^2)^2} &= \frac{4}{\sigma\pi^2}\left(\frac{\pi}{4}\log\frac{\sqrt{\pi\sigma}}{\epsilon} + \Gamma_{1/a}\right)\end{aligned} \tag{6.3.19}$$

with

$$\Gamma_a = \frac{\pi}{8}\log\frac{1}{a^2} - \frac{1}{2}\text{arctg}\,\frac{1}{a^2} - \frac{1}{2}\int_0^{1/a} \frac{\text{arctg}\, y}{y}\,dy;$$

and from (6.3.18) we conclude

$$\Delta_\epsilon \le -\frac{4\epsilon^2}{\sigma}(8\pi\Delta\gamma(0) + \Delta u_0(0))\log\frac{\sqrt{\sigma\pi}}{\epsilon} + O(\epsilon^2), \tag{6.3.20}$$

as $\epsilon \to 0$.

Recalling that $\Delta\gamma(0) = \frac{1}{|M|}$, from (6.3.13) and (6.3.20), we deduce (6.3.10). □

At this point, to establish Theorem 6.3.8, we shall use as a minimizing sequence for I, the family w_λ constructed in Section 4.4 of Chapert 4 (see Remark 4.4.28).

Thus for M in (6.3.1), u_0 satisfying (6.3.4), and λ large, we obtain a function w_λ satisfying:

$$\begin{cases} -\Delta w_\lambda = 8\pi\left(\dfrac{e^{w_0+u_\lambda}}{\int_M e^{u_0+w_\lambda}} - \dfrac{1}{|M|}\right) + f_\lambda \text{ in } M, \\ w_\lambda \in E, \end{cases} \tag{6.3.21}$$

for a suitable function $f_\lambda \in C(M)$ with $\int_M f_\lambda = 0$ and such that, as $\lambda \to \infty$:

$$||f_\lambda||_{L^1(M)} = O(b_\lambda), \quad ||\, f_\lambda \log|f_\lambda|\,||_{L^1(M)} = O(b_\lambda \log\lambda) \tag{6.3.22}$$

with

$$b_\lambda = \frac{1}{\lambda}\frac{\int_M e^{2(u_0+w_\lambda)}}{(\int_M e^{u_0+w_\lambda})^2} \to 0 \text{ and } \liminf_{\lambda\to+\infty} b_\lambda \log\lambda = 0. \tag{6.3.23}$$

In addition

$$I(w_\lambda) \to \inf_E I, \text{ as } \lambda \to +\infty, \tag{6.3.24}$$

and I attains its infimum on E if and only if w_λ is bounded in E uniformly in λ (see Lemma 4.4.27).

Notice in particular that if $w_{2,\lambda}$ is the *unique* solution for the problem

$$\begin{cases} -\Delta w_{2,\lambda} = f_\lambda \text{ in } M \\ w_{2,\lambda} \in E \end{cases}$$

then

$$||\nabla w_{2,\lambda}||_{\mathrm{L}^2(M)} + \max_M |w_{2,\lambda}| = O(b_\lambda \log \lambda).$$

Therefore, along a sequence, (whose existence is ensured by the second condition in (6.3.23))

$$\lambda_n \to +\infty \text{ such that } b_{\lambda_n} \log \lambda_n \to 0 \tag{6.3.25}$$

we see that

$$||\nabla w_{2,\lambda_n}||_{\mathrm{L}^2(M)} + \max_M |w_{2,\lambda_n}| \to 0 \text{ as } n \to +\infty. \tag{6.3.26}$$

Consequently, the new sequence

$$w_n = w_{2,\lambda_n} - w_{\lambda_n} \tag{6.3.27}$$

satisfies

$$\begin{cases} -\Delta w_n = 8\pi \left(\frac{h_n e^{w_n}}{\int_M h_n e^{w_n}} - \frac{1}{|M|} \right) \\ w_n \in E \end{cases} \tag{6.3.28}$$

where

$$h_n = e^{u_0 + w_{2,\lambda_n}} \to h = e^{u_0} \text{ uniformly in } M. \tag{6.3.29}$$

Furthermore,

$$\begin{cases} \frac{1}{2}||\nabla w_n||^2_{\mathrm{L}^2(M)} - 8\pi \log \fint_M h_n e^{w_n} \to \inf_E I, \\ I \text{ attains its infimum on } E \text{ if and only if the sequence } w_n \text{ is} \\ \text{uniformly bounded in } E. \end{cases} \tag{6.3.30}$$

We analyze what happens in the case

$$||\nabla w_n||^2_{\mathrm{L}^2(M)} \to +\infty, \text{ as } n \to +\infty. \tag{6.3.31}$$

In this direction we prove:

Lemma 6.3.11 *If (6.3.31) holds, then w_n admits a unique blow-up point $p_0 \in M$ such that $h(p) = e^{u_0(p)} > 0$, and*

$$\frac{h_n e^{w_n}}{\int_M h_n e^{w_n}} \to \delta_{p_0}, \quad \text{weakly in the sense of measure in } M \tag{6.3.32}$$

$$\max_D \left(w_n - \log \int_M h_n e^{w_n} \right) \to -\infty, \ \forall D \subset\subset M \setminus \{p_0\} \tag{6.3.33}$$

Proof. By virtue of (6.3.28), the condition (6.3.31) implies that

$$\max_M \left(w_n - \log \int_M h_n e^{w_n} \right) \to +\infty \text{ as } n \to \infty, \tag{6.3.34}$$

and so w_n must admit a blow-up point in M. We seek a characterization of such a blow-up point in order to ensure that it occurs away from the zero set of h.

To this purpose, recall that the functional I is coercive over the functions that satisfy conditions (6.2.16) in Lemma 6.2.7. Thus, there must exist a point $p_0 \in M$ such that

$$\frac{\int_{U_\delta(p_0)} e^{w_n}}{\int_M e^{w_n}} \to 1, \ \text{as } n \to \infty \tag{6.3.35}$$

for any small $\delta > 0$. Thus p_0 must coincide with a blow-up point for w_n, (possibly along a subsequence). We claim that $h(p_0) > 0$. To this purpose, for any small $\delta > 0$, we use (6.3.35) to deduce that

$$⨍_M h_n e^{w_n} \le \left(\max_{U_\delta(p_0)} h_n + o(1) \right) ⨍_M e^{w_n} = \left(\max_{U_\delta(p_0)} h + o(1) \right) ⨍_M e^{w_n},$$

for n sufficiently large. Therefore,

$$\begin{aligned} \inf_E I &= I(w_n) + o(1) \\ &= \frac{1}{2} ||\nabla w_n||^2_{L^2(M)} - 8\pi \log ⨍_M h_n e^{w_n} + o(1) \ge \frac{1}{2} ||\nabla w_n||^2_{L^2(M)} \\ &\quad -8\pi \log \left(⨍_M e^{w_n} \right) - 8\pi \log \left(\max_{U_\delta(p_0)} h + o(1) \right) + o(1), \ \text{as } n \to \infty. \end{aligned}$$

Thus, by means of the Moser–Trudinger inequality (2.4.17), we find a constant $C > 0$, such that

$$\log \left(\max_{U_\delta(p_0)} h \right) \ge -C,$$

for every $\delta > 0$.

Hence, by letting $\delta \to 0$ and using the continuity of h, we conclude that $h(p_0) > 0$ as claimed.

This information allows us to apply Proposition 5.7.64 and arrive at the desired conclusion. □

As a consequence of Lemma 6.3.11, we see that

$$\text{if } x_n \in M : \ w_n(x_n) = \max_M w_n. \tag{6.3.36}$$

Then

$$x_n \to p_0 \text{ and } \rho_n := w_n(x_n) - \log \int_M h_n e^{w_n} \to +\infty, \text{ as } n \to +\infty \tag{6.3.37}$$

(possibly along a subsequence).

Set

$$s_n = e^{-\frac{\rho_n}{2}} \to 0, \text{ as } n \to +\infty.$$

We consider the doubly periodic functions w_n and h_n extended (by periodicity) over $\mathbb{R}^2$ and define:

$$\xi_n(z) = w_n(x_n + s_n z) - w_n(x_n) - \frac{2\pi |s_n z|^2}{|M|}, \text{ for } z \in \Omega_n := \{x : s_n x \in \Omega\}. \tag{6.3.38}$$

Hence ξ_n satisfies:

$$\begin{cases} -\Delta \xi_n = U_n e^{\xi_n} \text{ in } \Omega_n, \\ \xi_n(0) = \max_{\Omega_n} \xi_n = 0, \\ \int_{\Omega_n} U_n e^{\xi_n} = 8\pi, \end{cases}$$

with

$$U_n(z) = 8\pi h_n(x_n + s_n z) e^{\frac{2\pi |s_n z|^2}{|M|}} \to 8\pi e^{u_0(p_0)} > 0, \tag{6.3.39}$$

uniformly in $C^0_{\text{loc}}(\mathbb{R}^2)$.

We can apply Lemma 5.4.21 (with $a = 0$ and $\mu = 8\pi e^{u_0(p_0)}$), together with Remark 5.4.22, to conclude that along a subsequence

$$\xi_n \to \xi \text{ uniformly in } C^0_{\text{loc}}(\mathbb{R}^2) \text{ with } \xi(z) = \log \left(\frac{1}{(1 + \pi \sigma |z|^2)^2} \right) \tag{6.3.40}$$

and with $\sigma = e^{u_0(p_0)}$. On the other hand, we may also write

$$\xi_n = u_n(s_n z) + 2 \log s_n,$$

where

$$u_n(x) = w_n(x_n + z) - \log \int_M h_n e^{w_n} - \frac{2\pi |z|^2}{|M|}, \ z \in \Omega$$

satisfies

$$\begin{cases} -\Delta u_n = V_n e^{u_n} & \text{in } \Omega \\ V_n e^{u_n} \to 8\pi \delta_{z=0}, & \text{weakly in the sense of measure in } \Omega \end{cases}$$

and

$$V_n(z) = 8\pi h_n(x_n + z) e^{\frac{2\pi|z|^2}{|M|}} \to 8\pi e^{u_o(p_0+z)+\frac{2\pi|z|^2}{|M|}} := V(z),$$

uniformly in Ω. Notice in particular that $V(0) = 8\pi e^{u_0(p_0)} > 0$.

From (6.3.32) and well-known elliptic estimates, we see that $w_n(z) \to 4G(z-p_0)$, uniformly in $C^0_{\text{loc}}(M \setminus p_0)$, with G the Green's function. Thus for u_n, we can also check

$$\max_{\partial\Omega} u_n - \min_{\partial\Omega} u_n \le C,$$

for a suitable constant $C > 0$.

Therefore, by virtue of Remark 5.6.53 (b), we are in a position to apply Lemma 5.6.52 (with $\alpha = 0$) for ξ_n and obtain:

Lemma 6.3.12 *Let ξ_n be given as in (6.3.38). For every $\varepsilon > 0$, there exist constants $R_\varepsilon > 0$, and $C_\varepsilon > 0$, and $n_\varepsilon \in \mathbb{N}$:*

$$\xi_n(z) \le (4-\varepsilon)\log\left(\frac{1}{|z|}\right) + C_\varepsilon,\ |z| \ge R_\varepsilon \tag{6.3.41}$$

for $n \ge n_\varepsilon$.

By the asymptotic decay properties in (6.3.41), we obtain:

Proposition 6.3.13 *Let w_n in (6.3.27) satisfy (6.3.31) and let p_0 be given as in Lemma 6.3.11. Then $u_0(p_0) = \max_M u_0$ and*

$$\begin{aligned} \inf_E I &= \lim_{n\to\infty} \left(\frac{1}{2}||\nabla w_n||^2_{L^2(M)} - 8\pi \log ⨍_M h_n e^{w_n}\right) \\ &= -8\pi\Big(4\pi\gamma(0) + u_0(p_0) + \log(\pi|M|^2) + 1\Big). \end{aligned} \tag{6.3.42}$$

Proof. Observe that

$$||\nabla w_n||^2_{L^2(M)} = 8\pi \frac{\int_M h_n e^{w_n} w_n}{\int_M h_n e^{w_n}}.$$

Thus,

$$\begin{aligned} ||\nabla w_n||^2_{L^2(M)} &= 8\pi \frac{\int_M e^{u_0+w_n} w_n}{\int_M e^{u_0+w_n}} \\ &= 8\pi w_n(x_n) + 8\pi \int_M h_n e^{(w_n - \log \int_M h_n e^{w_n})}(w_n - w_n(x_n)) + o(1) \\ &= 8\pi w_n(x_n) + \int_{\Omega_n} U_n e^{\xi_n}\xi_n + o(1). \end{aligned}$$

By (6.3.39) and the decay estimate (6.3.41), we can use the dominated convergence theorem to pass to the limit into the integral above and obtain:

$$\int_{\Omega_n} U_n e^{\xi_n} \xi_n \to -8\pi\sigma \int_{\mathbb{R}^2} \frac{\log(1+\pi\sigma|z|^2)^2}{(1+\pi\sigma|z|^2)^2}, \text{ with } \sigma = e^{u_0(p_0)}. \tag{6.3.43}$$

Using the identity: $\sigma \int_{\mathbb{R}^2} \frac{\log(1+\pi\sigma|z|^2)}{(1+\pi\sigma|z|^2)^2} = 1$, we conclude that

$$||\nabla w_n||^2_{L^2(M)} = 8\pi\, w_n(x_n) - 16\pi + o(1), \text{ as } n \to \infty.$$

Consequently as $n \to \infty$,

$$\frac{1}{2}||\nabla w_n||^2_{L^2(M)} - 8\pi \log\left(⨍_M h_n e^{w_n}\right) = 4\pi\left(w_n(x_n) - 2\log\left(\int_M h_n e^{w_n}\right)\right)$$
$$-8\pi + 8\pi \log|M| + o(1).$$

We use Green's representation formula for w_n to find

$$\begin{aligned} w_n(x_n) &= \frac{1}{2}\int_M G(x_n, y)\frac{h_n e^{w_n}}{\int_M h_n e^{w_n}} = \frac{1}{2\pi}\int_{\Omega_n} G(x_n, x_n + s_n y) U_n(y) e^{\xi_n(y)} \\ &= \frac{1}{2\pi}\int_{\Omega_n} \log\left(\frac{1}{|s_n y|}\right) U_n(y) e^{\xi_n(y)} + \int_{\Omega} \gamma\,(s_n y) U_n(y) e^{\xi_n(y)} \\ &= 4\log\frac{1}{s_n} + \frac{1}{2\pi}\int_{\Omega_n} \log\left(\frac{1}{|y|}\right) U_n(y) e^{\xi_n(y)} + \int_{\Omega_n} \gamma\,(s_n y) U_n(y) e^{\xi_n(y)}. \end{aligned}$$

As before, by (6.3.41) we can justify the passage to the limit into the integrals above and conclude,

$$\begin{aligned} w_n(x_n) - 2\log\int_M h_n e^{w_n} &= 4\log\frac{1}{s_n} - w_n(x_n) = 4\int_{\mathbb{R}^2} \frac{\sigma\log|y|}{(1+\sigma\pi|y|^2)^2} \\ &\quad -8\pi\gamma\,(0) + o(1), \text{ as } n \to \infty, (\sigma = e^{u_0(p_0)}). \end{aligned} \tag{6.3.44}$$

In view of the identity $2\int \frac{\sigma\log|y|}{(1+\sigma\pi|y|^2)^2} = -\log(\pi\sigma)$, we conclude that

$$I(w_n) = -8\pi\left(4\pi\gamma\,(0) + u_0(p_0) + \log\left(\frac{\pi}{|M|}\right) + 1\right) + o(1),$$

for large n. Hence, by letting $n \to +\infty$ we find:

$$\inf_E I = -8\pi\left(4\pi\gamma\,(0) + u_0(p_0) + \log\left(\frac{\pi}{|M|}\right) + 1\right). \tag{6.3.45}$$

Comparing (6.3.8) with (6.3.45), we deduce that necessarily

$$u_0(p_0) = \max_M u_0, \tag{6.3.46}$$

and the proof is completed. □

Proof of Theorem 6.3.8. Accordingly to Proposition 6.3.13 and the properties in (6.3.30), if I does not attain its infimum in E, then necessarily

$$\inf_E I = -8\pi\left(4\pi\gamma(0) + \max_M u_0 + \log\left(\frac{\pi}{|M|}\right) + 1\right). \tag{6.3.47}$$

On the other hand, by virtue of Proposition 6.3.10, when (6.3.6) holds we can ensure that

$$\inf_E I < -8\pi\left(4\pi\gamma(0) + \max_M u_0 + \log\left(\frac{\pi}{|M|}\right) + 1\right),$$

whence I must attain its infimum in E as claimed. □

6.4 The proof of Theorem 4.4.29

The analysis of the previous section (with u_0 in (4.1.3) and $N = 2$), enables us to complete the proof of Theorem 4.4.29. Recall that by virtue of Lemma 4.4.27, we have to analyze only the case where the functional $I_{\mu=8\pi}$ $(= I)$ does not attain its infimum in E, or equivalently when the family w_λ $(= w_\lambda^-)$ in (4.4.26), (4.4.27), and (4.4.28) satisfies

$$||w_\lambda||_E \to +\infty, \text{ as } \lambda \to +\infty. \tag{6.4.1}$$

As in the proof of Theorem 6.3.8, in this situation we are able to find a sequence $\lambda_n \to +\infty$, such that $w_n = w_{\lambda_n} - w_{2,\lambda_n}$ (where $w_{2,\lambda}$ is defined in (4.4.38)) satisfies: (6.3.32), (6.3.33), and (6.3.46). Recalling (4.4.22) and (4.4.19) and that in Theorem 4.4.29 we take: $v_{2,\lambda} = w_\lambda + d_\lambda$. Then we can check that, for every doubly periodic continuous test function φ, there holds

$$\begin{aligned}\lambda_n \int_\Omega e^{u_0+v_{2,\lambda_n}}(1 - e^{u_0+v_{2,\lambda_n}})\varphi &= \lambda_n d_{\lambda_n} \int_\Omega e^{u_0+w_{\lambda_n}}(1 - d_{\lambda_n} e^{u_0+w_{\lambda_n}})\varphi \\ &= 8\pi \int_\Omega \frac{e^{u_0+w_{\lambda_n}}}{\int_\Omega e^{u_0+w_{\lambda_n}}}\varphi - \frac{8\pi}{\lambda_n}\int_\Omega \frac{e^{2(u_0+w_{\lambda_n})}}{\left(\int_\Omega e^{u_0+w_{\lambda_n}}\right)^2}\varphi + o(1) \\ &= 8\pi \int_\Omega \frac{h_n e^{w_n}}{\int_\Omega h_n e^{w_n}}\varphi + o(1) \to 8\pi\varphi(p_0), \text{ as } n \to \infty\end{aligned}$$

and $u_0(p_0) = \max_M u_0$; and so claim (4.4.45) is established.

It remains to verify the uniform limit,

$$\max_{\overline{\Omega}} e^{u_0+v_{2,\lambda}} \to 0, \text{ as } \lambda \to +\infty. \tag{6.4.2}$$

Notice that according to (4.4.22), the limit in (6.4.2) is equivalent to

$$\max_{\overline{\Omega}}\ e^{u_0+w_\lambda-\log\int_\Omega e^{u_0+w_\lambda}-\log\lambda} \to 0, \text{ as } \lambda \to +\infty. \tag{6.4.3}$$

To this end, notice that (6.4.3) certainly holds along a sequence λ_n chosen to satisfy (6.3.25).

Indeed for such sequence λ_n, we can use the analysis of the previous section for $w_n = w_{\lambda_n} - w_{2,\lambda_n}$ to find

$$s_n^2 \frac{\int_\Omega e^{2(u_0+w_{\lambda_n})}}{\left(\int_\Omega e^{u_0+w_{\lambda_n}}\right)^2} = s_n^2 \frac{\int_\Omega h_n^2 e^{2w_n}}{\left(\int_\Omega h_n e^{w_n}\right)^2} = s_n^2 \int_\Omega h_n^2 e^{2(w_n - \log \int_\Omega h_n e^{w_n})} = \int_{\Omega_n} U_n^2 e^{2\xi_n}$$
$$\to (8\pi)^2 \sigma^2 \int_{\mathbb{R}^2} \frac{1}{(1+\sigma\pi|z|^2)^4} = \frac{1}{3}(8\pi)^2\sigma, \text{ as } n \to \infty,$$

where we recall that $\sigma = e^{\max_{\bar\Omega} u_0}$, $s_n = e^{-\frac{1}{2}(\max_{\bar\Omega} w_n - \log \int_\Omega h_n e^{w_n})}$, ξ_n and U_n are defined in (6.3.38) and (6.3.39) respectively, and the limit of the latter integral is justified by (6.3.40) and (6.3.41). Consequently, recalling that

$$\int_\Omega \frac{e^{2(u_0+w_n)}}{\left(\int_\Omega e^{(u_0+w_n)}\right)^2} = o(\lambda_n), \quad \text{(see (4.4.19))}$$

we deduce $\frac{1}{s_n^2 \lambda_n} \to 0$, as $n \to \infty$; that is

$$e^{\max_{\bar\Omega}\left(w_n - \log \int_\Omega h_n e^{w_n}\right) - \log \lambda_n} \to 0, \text{ as } n \to \infty. \tag{6.4.4}$$

Since $e^{u_0} \in L^\infty(M)$, (6.4.3) follows for $\lambda = \lambda_n$.

Thus, to establish (6.4.3) it suffices to show that in fact along *any* sequence $\lambda_n \to +\infty$, property (6.3.25) holds. In other words, the following stronger version of Lemma 4.4.27 holds:

Proposition 6.4.14

$$\lim_{\lambda \to \infty} b_\lambda \log \lambda = 0, \tag{6.4.5}$$

where

$$b_\lambda = \frac{1}{\lambda} \frac{\int_\Omega e^{2(u_0+w_\lambda)}}{\left(\int_\Omega e^{u_0+w_\lambda}\right)^2},$$

and w_λ, *is defined by* (4.4.26), (4.4.27), *and* (4.4.28).

Proof. To simplify notation, we perform a translation of the coordinates so that $p_0 = 0 \in \Omega$ and $u_0(0) = \max_M u_0$. We recall that u_0 is defined by (4.1.3) with $N = 2$; therefore u_0 attains its maximum value at a point away from the vortex points. We thus have: $-\Delta u_0(0) + \frac{8\pi}{|\Omega|} = 0$. Therefore, for $\varepsilon > 0$ and w_ε in (6.3.9), and by the arguments of Proposition 6.3.10, we find:

$$\begin{aligned} I(w_\varepsilon) &= \frac{1}{2}\int_\Omega |\nabla w_\varepsilon|^2 - 8\pi \log ⨍_\Omega e^{u_0+w_\varepsilon} \\ &\le -8\pi\left(4\pi\gamma(0) + u_0(0) + \log\left(\frac{\pi}{|\Omega|}\right) + 1\right) + O(\varepsilon^2) \text{ as } \varepsilon \to 0. \end{aligned} \tag{6.4.6}$$

To obtain (6.4.5), we argue by contradiction and assume that there exist a sequence $\lambda_n \to +\infty$ and a constant $\beta > 0$ such that for $w_n = w_{\lambda_n}$, there holds

$$\frac{\log \lambda_n}{\lambda_n} \frac{\int_\Omega e^{2(u_0+w_n)}}{\left(\int_\Omega e^{u_0+w_n}\right)^2} \geq \frac{\beta}{(8\pi)^2}, \text{ for } n \text{ large}. \tag{6.4.7}$$

Clearly (6.4.7) implies that the sequence w_n is unbounded in E. Therefore, the functional $I_{\mu=8\pi} = I$ in (6.3.5) cannot attain its infimum in E, and so,

$$\inf_E I = -8\pi\left(4\pi\gamma(0) + u_0(0) + \log\left(\frac{\pi}{|\Omega|}\right) + 1\right). \tag{6.4.8}$$

Now recall that w_λ is defined by the extremal property: $f_\lambda(w_\lambda) = \inf_{A_\lambda} f_\lambda$, where f_λ is defined in (4.4.12) and A_λ is given in (4.4.5). We wish to use (6.4.6) together with (6.4.8) to evaluate the functional $f_{\lambda=\lambda_n}$ over w_ε in (6.3.9), for suitable $\varepsilon > 0$. To this purpose, notice that for $x \in \Omega_\varepsilon = \{z : \varepsilon z \in \Omega\}$, we have

$$\begin{aligned} w_\varepsilon(\varepsilon x) &= \frac{4}{A_\varepsilon}\int_{\Omega_\varepsilon} \log\left(\frac{1}{\varepsilon|x-y|}\right)\rho_\varepsilon(y)\,dy + \frac{8\pi}{A_\varepsilon}\int_{\Omega_\varepsilon}\gamma(\varepsilon(x-y))\rho_\varepsilon(y)\,dy \\ &= 4\log\frac{1}{\varepsilon} + 4\int_{\mathbb{R}^2}\log\left(\frac{1}{|x-y|}\right)\frac{\sigma}{(1+\pi\sigma|y|^2)^2}\,dy + 8\pi\gamma(0) + R_\varepsilon(x), \end{aligned}$$

where ρ_ε and A_ε are given in (6.3.11), $\sigma = e^{\max_{\bar\Omega} u_0}$ and where the remainder $R_\varepsilon \to 0$ a.e., as $\varepsilon \to 0$. Since $\beta_\epsilon(x) = \frac{e^{u_0(0)} - e^{u_0(\epsilon y)}}{(1+\pi\sigma|y|^2)^2}$ satisfies $\int_{\Omega_\epsilon}\beta_\epsilon(x) = O(\epsilon^2 \log\frac{1}{\epsilon})$ (see (6.3.15)), we may estimate R_ϵ as follows:

$$\begin{aligned} R_\epsilon(x) &= 4\int_{\Omega_\epsilon}\log|x-y|\,\beta_\epsilon(y)dy + 4\int_{\mathbb{R}^2\setminus\Omega_\epsilon}\log\frac{1}{|x-y|}\frac{e^{u_0(0)}}{(1+\pi\sigma|y|^2)^2}dy \\ &\quad + \frac{8\pi}{A_\epsilon}\int_{\Omega_\epsilon}(\gamma(\epsilon(x-y)) - \gamma(0))\rho_\epsilon(y)dy \leq 4\int_{\Omega_\epsilon\cap\{|y|\leq|x|+1\}}\log|x-y|\,\beta_\epsilon(y)dy \\ &\quad + 4\int_{\Omega_\epsilon\cap\{|x|\leq|y|-1\}}\log|x-y|\,\beta_\epsilon(y)dy + 4e^{u_0(0)}\int_{\{|x-y|\leq 1\}}\log\frac{1}{|x-y|}dy \\ &\quad + 16\pi\max_\Omega|\gamma| \leq 4\log(2|x|+1)\int_{\Omega_\epsilon}\beta_\epsilon(y)dy + 8e^{\max_\Omega u_0}\int_{\mathbb{R}^2}\frac{\log 2|y|}{(1+\pi\sigma|y|^2)^2}dy \\ &\quad + 2\pi e^{u_0(0)} + 16\pi\max_\Omega|\gamma| \leq O(\epsilon^2\log\frac{1}{\epsilon})\log(2|x|+1) + C, \text{ as } \epsilon \to 0, \end{aligned} \tag{6.4.9}$$

with $C > 0$ a suitable constant.

Thus, in view of the identity: $4\int\log\frac{1}{|x-y|}\frac{\sigma}{(1+\pi\sigma|y|^2)^2} = 2\log\left(\frac{\pi\sigma}{1+\sigma\pi|x|^2}\right)$, we have

$$w_\epsilon(\epsilon x) = \log\frac{1}{\epsilon^4} + 2\log\left(\frac{\pi\sigma}{1+\sigma\pi|x|^2}\right) + 8\pi\gamma(0) + R_\epsilon(x), \text{ as } \epsilon \to 0. \tag{6.4.10}$$

From (6.4.9) it follows in particular that

$$w_\epsilon(\epsilon x) \leq \log\frac{1}{\epsilon^4} + \log\frac{1}{(1+|x|^2)^{2-\alpha}} + C, \tag{6.4.11}$$

for fixed $\alpha \in (0, 2)$, a suitable constant $C > 0$ and $\epsilon > 0$ sufficiently small.

Hence,

$$\int_\Omega e^{2(u_0(x)+w_\epsilon(x))}dx = \epsilon^2 \int_{\Omega_\epsilon} e^{2(u_0(\epsilon x)+w_\epsilon(\epsilon x))}dx$$
$$= \epsilon^2 \int_{\Omega_\epsilon} e^{2(u_0(\epsilon x)+\log \frac{1}{\epsilon^4}+2\ \log(\frac{\pi\sigma}{1+\sigma\pi|x|^2})+8\pi\gamma(0)+R_\epsilon(x))}$$
$$= \frac{1}{\epsilon^6}\left(e^{4\log\ \pi\sigma+16\pi\gamma(0)}\int \frac{\sigma^2}{(1+\sigma\pi|x|^2)^4}dx + o(1)\right),$$

as $\epsilon \to 0$. Here we have used estimate (6.4.11) to justify the passage to the limit into the integral sign.

Analogously,

$$\int_\Omega e^{u_0(x)+w_\epsilon(x)}dx = \frac{1}{\epsilon^2}\left(e^{2\log\ \pi\sigma+8\pi\gamma(0)}\int \frac{\sigma}{(1+\sigma\pi|x|^2)^2}dx + o(1)\right),$$

as $\epsilon \to 0$. Therefore:

$$\frac{\int_\Omega e^{2(u_0(x)+w_\epsilon(x))}}{(\int_\Omega e^{u_0(x)+w_\epsilon(x)})^2} = \frac{1}{\epsilon^2}\left(\frac{\int \frac{e^{2u_0(0)}}{(1+\sigma\pi|x|^2)^4}}{(\int \frac{e^{u_0(0)}}{(1+\sigma\pi|x|^2)^2})^2} + o(1)\right)$$
$$= \frac{1}{\epsilon^2}\left(\frac{\sigma}{3} + o(1)\right), \quad \text{as } \epsilon \to 0. \tag{6.4.12}$$

Choose $\epsilon_n > 0$ such that

$$\frac{1}{\epsilon_n^2}\left(\frac{1}{\lambda_n}(8\pi)^2\frac{\sigma}{3}\right) = \frac{\beta}{2}\frac{1}{\log\ \lambda_n} \tag{6.4.13}$$

and set $w_n^* = w_{\epsilon_n}$.

Clearly $\epsilon_n \to 0$ as $n \to +\infty$, and in view of (6.4.12) and (6.4.13), we can easily check that $w_n^* \in \mathcal{A}_{\lambda_n}$ for large n. From (6.4.6), (6.4.8), and $w_n = w_{\lambda_n}$, we have:

$$O(\epsilon_n^2) \geq I(w_n^*) - I(w_n) = f_{\lambda_n}(w_n^*) - f_{\lambda_n}(w_n)$$
$$-8\pi\ \psi\left(1+\sqrt{1-\frac{32\pi}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2}}\right)$$
$$+8\pi\ \psi\left(1+\sqrt{1-\frac{32\pi}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n^*)}}{(\int_\Omega e^{u_0+w_n^*})^2}}\right)$$
$$\geq \frac{(16\pi)^2}{2\lambda_n}\left(\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2} - \frac{\int_\Omega e^{2(u_0+w_n^*)}}{(\int_\Omega e^{u_0+w_n^*})^2}\right)$$
$$\times \int_0^1 \frac{dt}{\left(1+\sqrt{1-\frac{32\pi}{\lambda_n}\left(t\frac{\int_\Omega e^{2(u_0+w_n^*)}}{(\int_\Omega e^{u_0+w_n^*})^2}+(1-t)\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2}\right)}\right)^2}$$
$$= \frac{(16\pi)^2}{2\lambda_n}\left(\frac{1}{4}+o(1)\right)\left(\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2} - \frac{\int_\Omega e^{2(u_0+w_n^*)}}{(\int_\Omega e^{u_0+w_n^*})^2}\right).$$

That is,

$$\frac{1}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2} \le \frac{1}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n^*)}}{(\int_\Omega e^{u_0+w_n^*})^2} + O(\epsilon_n^2) = \frac{1}{\lambda_n \epsilon_n^2}\frac{\sigma}{3} + o\left(\frac{1}{\log\, \lambda_n}\right).$$

Therefore, for large n,

$$\begin{aligned}\frac{\beta}{\log \lambda_n} \le a_{\lambda_n} &= \frac{(16\pi)^2}{\left(1+\sqrt{1-\frac{32\pi}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2}}\right)^2}\frac{1}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2} = \frac{(8\pi)^2}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2} \\ &\quad + \frac{(16\pi)^2}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2}\left(\frac{1}{(1+\sqrt{1-\frac{32\pi}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2}})^2} - \frac{1}{4}\right) \\ &= \frac{(8\pi)^2}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2}\left(1+\frac{16\pi}{\lambda_n}\frac{\int_\Omega e^{2(u_0+w_n)}}{(\int_\Omega e^{u_0+w_n})^2}(1+o(1)\right) \\ &\le \frac{(8\pi)^2}{\lambda_n}\frac{1}{\epsilon_n^2}\frac{\sigma}{3} + o\left(\frac{1}{\log\, \lambda_n}\right).\end{aligned}$$

This is clearly impossible by the choice of ϵ_n in (6.4.13), and (6.4.5) is established.

As a consequence of Proposition 6.4.14, we see that (6.4.4) holds along any sequence $\lambda_n \to +\infty$. Therefore (6.4.3) is valid, and the proof of Theorem 4.4.29 is completed. □

6.5 Final remarks and open problems

In concluding, we mention that it would be extremely useful to find extensions of the formula (2.5.10) to evaluate the degree d_μ of the Fredholm map associated to problem (6.1.1), (6.1.2), beyond the case $\mu \in (0, 16\pi) \setminus \{8\pi\}$ as given in (6.1.6) (cf. [ChLW]).

Since for problem (6.1.1), (6.1.2) existence results are not yet available when $\mu > 16\pi$, already it would be useful to know the surface M, for which we can establish that $d_\mu \neq 0$ for large μ.

As we have seen, any information about d_μ could be immediately turned into an information for the degree of the operator associated to problem (6.1.7), which enters into the construction of electroweak vortex configurations, as we shall explain in the next chapter.

Also, as for the previous chapter, it would be relevant to extend the analysis above to cover systems. Needless to say, one of the main difficulties associated with such an extension is the fact that crucial tools, such as the maximum principle, are no longer available for systems.

Already for the $SU(n+1)$-Toda system, which we have seen to arise in a natural way in the study of non-abelian Chern–Simons vortices, such difficulties are not easy to overcome, even if we neglect the presence of the Dirac measures.

In this context we have already mentioned the contributions in [CSW], [W], [JoW1], [JoW2], [LN], [ChOS], [MN], and [SW1], [SW2]. Recently, Jost–Lin–Wang in [JoLW] have established suitable extensions of some of the results given above by a deeper understanding of the "bubbling" phenomenon for solutions of the $SU(n+1)$-Toda system.

7

Selfdual Electroweak Vortices and Strings

7.1 Introduction

In Section 1.4, we reviewed how to attain selfduality for the $SU(2)\times U(1)$-electroweak theory of Glashow–Salam–Weinberg [La] described by the Lagrangean density (1.4.15).

Recall that, for the unitary gauge variables given in (1.4.17)–(1.4.20) and with the help of the vortex ansatz (1.4.26)–(1.4.29), we may formulate the theory in terms of a complex valued massive field W, a scalar field φ, and two (real) 2-vector fields, $P = (P_j)_{j=1,2}$ and $Z = (Z_j)_{j=1,2}$, which are assumed to depend only on the (x^1, x^2)-variables. The massive field W is weakly coupled to P and Z through the covariant derivative:

$$D_j W = \partial_j W - ig(P_j \sin\theta + Z_j \cos\theta)W,\ \ j = 1, 2 \tag{7.1.1}$$

where g is the $SU(2)$-coupling constant and $\theta \in (0, \frac{\pi}{2})$ is the Weinberg mixing angle that relates to the $U(1)$-coupling constant g_* by means of the identity:

$$\cos\theta = \frac{g}{(g^2 + g_*^2)^{\frac{1}{2}}}.$$

In this way the expression for the corresponding energy density $\mathcal{E}$ takes the form

$$\begin{aligned}\mathcal{E} = {}& |D_1 W + i D_2 W|^2 + \frac{1}{2}\left(P_{12} - \frac{g}{2\sin\theta}\varphi_0^2 - 2g\sin\theta|W|^2\right)^2 \\ & + \frac{1}{2}\left(Z_{12} - \frac{g}{2\cos\theta}\left(\varphi^2 - \varphi_0^2\right)^2 - 2g\cos\theta|W|^2\right)^2 + \left(\frac{g}{2\cos\theta}\varphi Z_j + \varepsilon_{jk}\partial_k\varphi\right)^2 \\ & - \frac{g^2}{8\sin^2\theta}\varphi_0^4 + \left(\lambda - \frac{g^2}{8\cos^2\theta}\right)\left(\varphi^2 - \varphi_0^2\right)^2 - \frac{g\varphi_0^2}{2\sin\theta} Z_{12} + \frac{g\varphi_0^2}{2\sin\theta} P_{12} \\ & - \frac{g}{2\cos\theta}\partial_k\left(\varepsilon_{jk} Z_j \varphi^2\right),\end{aligned} \tag{7.1.2}$$

where as usual, $Z_{12} = \partial_1 Z_2 - \partial_2 Z_1$ and $P_{12} = \partial_1 P_2 - \partial_2 P_1$ denote the "curl" of the vector field Z and P, respectively, and ε_{jk} denotes the total antisymmetric symbol with $\varepsilon_{12} = 1$.

Thus, in the "critical" coupling,

$$\lambda = \frac{g^2}{8\cos^2\theta}, \tag{7.1.3}$$

we deduce the following *selfdual equations*:

$$\begin{cases} D_1 W + i D_2 W = 0, \\ P_{12} = \dfrac{g}{2\sin\theta}(\varphi^2 - \varphi_0^2)^2 + 2g\cos\theta|W|^2, \\ Z_j = -\dfrac{2\cos\theta}{g}\varepsilon_{jk}\partial_k \log\varphi, \;\; j = 1, 2, \end{cases} \tag{7.1.4}$$

whose solutions minimize (7.1.2), whenever we satisfy appropriate boundary conditions to neglect the divergence terms.

In this Chapter, we shall be interested in establishing rigorous existence results for (7.1.4), in the planar case under a suitable decay assumption at infinity, and in the periodic case, under the 'tHooft periodic boundary condition. We refer to [AO1], [AO2], [AO3], [CM], [V1], [Y5], and [Y7], for other results on electroweak vortex-like solutions.

The planar solutions of (7.1.4) over $\mathbb{R}^2$ have been established first by Spruck–Yang in [SY1] by a shooting method, and more recently by Chae–Tarantello in [ChT1] by a perturbation approach similar in spirit to that introduced by Chae–Imanuvilov [ChI1] in the study of non-topological Chern–Simons vortices (discussed in Chapter 3). The constructions in [SY1] and [ChT1] yield to different classes of solutions for (7.1.4), distinguished according to their asymptotic behavior at infinity. It is not excluded that yet other type of solutions may exist. From (7.1.2), we see that solutions of (7.1.4) over $\mathbb{R}^2$ carry *infinite* energy, a fact that on one hand justifies their abundance, but on the other hand makes one wonder about their physical interest. Thus, to treat a problem with a more definite physical flavor, we shall focus on planar electroweak strings, where we also take into account the effect of gravity by coupling the electroweak equations with Einstein's equations.

We consider a model where strings are parallel to the x^3-direction, and where we take gravitational metrics to vary in the class:

$$ds^2 = (dx^0)^2 - (dx^3)^2 - e^\eta\left(\left(dx^1\right)^2 - \left(dx^2\right)^2\right). \tag{7.1.5}$$

The conformal factor $\eta = \eta(x^1, x^2)$ is coupled to the remaining electroweak variables through Einstein's equations.

Correspondingly, in the "critical" coupling (7.1.3), the *selfdual string equations* become:

$$\begin{cases} D_1 W + i D_2 W = 0, \\ P_{12} = \dfrac{g}{2\sin\theta}\varphi_0^2 e^\eta + 2g\sin\theta|W|^2, \\ Z_{12} = \dfrac{g}{2\cos\theta}\left(\varphi^2 - \varphi_0^2\right)e^\eta + 2g\cos\theta|W|^2, \\ Z_j = -\dfrac{2\cos\theta}{g}\varepsilon_{jk}\partial_k \log\varphi, \ \ j = 1, 2, \end{cases} \tag{7.1.6}$$

to be coupled with Einstein's equations, that under (7.1.5) reduce to

$$-\Delta\eta = 8\pi G\left(\frac{g\varphi_0^2}{\sin\theta}P_{12} + \frac{g}{\cos\theta}\left(\varphi^2 - \varphi_0^2\right)Z_{12} + 4|\nabla\varphi|^2\right), \tag{7.1.7}$$

with Newton's gravitational constant $G > 0$.

Recalling that the electroweak-string energy density is given by

$$\mathcal{E} = \frac{g^2\varphi_0^4}{8\sin^2\theta} + \frac{g^2}{4\cos^2\theta}\left(\varphi^2 - \varphi_0^2\right)^2 + g^2\varphi^2|W|^2 e^{-\eta} + 2e^{-\eta}|\nabla\varphi|^2, \tag{7.1.8}$$

we deduce (7.1.7) by observing that the Gauss curvature $K_\eta = -\frac{1}{2}e^{-\eta}\Delta\eta$ relative to the Riemann surface $(\mathbb{R}^2, e^\eta\delta_{j,k})$ satisfies the relation

$$K_\eta = 8\pi G\mathcal{E} + \Lambda, \tag{7.1.9}$$

where the cosmological constant Λ is fixed by Einstein's equations as follows:

$$\Lambda = \frac{\pi G g^2\varphi_0^4}{\sin^2\theta}. \tag{7.1.10}$$

For the corresponding selfgravitating electroweak strings, a meaningful physical property would be that the Riemann surface $(\mathbb{R}^2, e^\eta\delta_{j,k})$ carries *finite total curvature*; that is,

$$\int_{\mathbb{R}^2} K_\eta e^\eta < +\infty. \tag{7.1.11}$$

From (7.1.8) and (7.1.9), we see that (7.1.11) is actually equivalent to requiring that $e^\eta \in \mathrm{L}^1(\mathbb{R}^2)$ and that we satisfy the *finite total energy* condition with respect to the volume element of $(\mathbb{R}^2, e^\eta\delta_{j,k})$ as follows:

$$\int_{\mathbb{R}^2} \mathcal{E}e^\eta < +\infty. \tag{7.1.12}$$

For further details, we refer to [Y1] where actually the existence of solutions for problem (7.1.6), (7.1.7), (7.1.11) is listed as an *open* problem, which only recently has been successfully tackled by Chae–Tarantello in [ChT2].

On the contrary, more is known about cosmic strings relative to the coupling of Einstein's equations with other gauge fields theories such as the Maxwell–Higgs theory, the Chern–Simons theory, etc. (cf. [EGH], [Kb], and [VS]). In this respect, we mention the contributions in [Lint], [V2], [EN], [CG], [CHMcLY], [Y2], [Y4], [Ch4], [ChCh1]; and we refer to [Y1] for a more detailed discussion and additional references.

Clearly, if we neglect the gravitational effect by taking $G \equiv 0$ and $\eta \equiv 0$, then (7.1.6)–(7.1.7) reduce to (7.1.4). In fact, it is possible to follow the construction indicated in [ChT1] for electroweak vortices in order to obtain electroweak strings with the desired finite curvature and, energy property (7.1.11) and (7.1.12).

We present in details the construction of [ChT2] in the following section.

7.2 Planar selfgravitating electroweak strings

We devote this section to constructing a family of planar selfgravitating electroweak strings, corresponding to solutions for (7.1.6)–(7.1.7) and which satisfy the finite total curvature and energy conditions, namely, (7.1.11) and (7.1.12), respectively.

More precisely, we establish the following:

Theorem 7.2.1 *([ChT2]) Let $N \in \mathbb{N}$ satisfy:*

$$N + 1 < \frac{\sin^2 \theta}{4\pi G \varphi_0^2}. \tag{7.2.1}$$

For any given set of points $\{z_1, \dots, z_N\} \subset \mathbb{R}^2$ (repeated according to their multiplicity), there exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, we find a selfgravitating electroweak string $(W^\varepsilon, \varphi^\varepsilon, P^\varepsilon, Z^\varepsilon, \eta^\varepsilon)$ solution of (7.1.6)–(7.1.7), satisfying the finite curvature and energy conditions (7.1.11) and (7.1.12) with W^ε vanishing exactly at the points $\{z_1, \dots, z_N\}$ according to their multiplicity.

To obtain Theorem 7.2.1, we shall use (2.1.22) and (2.1.23) to reduce our problem to the search for solutions to the elliptic system (2.1.26) (derived in Section 2.1 of Chapter 2) formulated in terms of the variables (u, v, η), where:

$$|W|^2 = e^u \text{ and } e^v = \varphi^2.$$

More precisely, we shall analyze in $\mathbb{R}^2$, the following elliptic system:

$$\begin{cases} -\Delta u = g^2 e^{v+\eta} + 4g^2 e^u - 4\pi \sum_{j=1}^{N} \delta_{z_j}, \\ \Delta v = \dfrac{g^2}{2\cos^2\theta}\left(e^v - \varphi_0^2\right) e^\eta + 2g^2 e^u, \\ -\Delta \eta = 4\pi G g^2 e^\eta \left(\dfrac{\left(e^v - \varphi_0^2\right)^2}{\cos^2\theta} + \dfrac{\varphi_0^4}{\sin^2\theta} \right) + 16\pi G g^2 e^{u+v} + 8\pi G |\nabla v|^2 e^v. \end{cases} \tag{7.2.2}$$

Notice that, the given points $\{z_1, \dots, z_N\}$ (repeated with multiplicity) correspond to the zeroes of the massive field W which is given as follows:

$$W(z) = \exp\left(\frac{u}{2} + i \sum_{k=1}^{N} \arg(z - z_k) \right).$$

Recalling (1.4.18) and (1.4.19) we recover the remaining variables by means of the relations in (2.1.23) and (2.1.24).

We are going to attack problem (7.2.2) by a perturbation technique inspired by [ChI1] (see Section 3.4 of Chapter 3).

Therefore, the first goal will be to interpret the elliptic system (7.2.2) as a perturbation of a given Liouville-type operator. Due to its conformal invariance such a Liouville operator admits some degeneracies. Fortunately, it is possible to exploit the structure of the perturbation term in order to restore an invertibility property for the "perturbed" operator in a suitable functional space. This will allow us to use the Implicit Function theorem and to obtain a solution whose behaviour at infinity we can control rather well. In this way, we can also check the validity of (7.1.11) and (7.1.12).

To this purpose, we transform (7.2.2) to an equivalent system as follows. We multiply the second equation of (7.2.2) by e^{υ} and use the identity $\Delta e^{\upsilon} = e^{\upsilon}\Delta\upsilon + |\nabla\upsilon|^2 e^{\upsilon}$ to deduce:

$$\Delta e^{\upsilon} = \frac{g^2}{2\cos^2\theta}\left(e^{\upsilon} - \varphi_0^2\right)e^{\eta+\upsilon} + 2g^2 e^{u+\upsilon} + |\nabla\upsilon|^2 e^{\upsilon}.$$

Next we multiply the equation above by $8\pi G$ and add the result to the third equation in (7.2.2) to find:

$$\Delta(\eta + 8\pi G e^{\upsilon}) = -4\pi G g^2\varphi_0^4\left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right)e^{\eta} + \frac{4\pi G g^2\varphi_0^2}{\cos^2\theta}e^{\eta+\upsilon}.$$

Thus, letting

$$\lambda_1 = 4g^2, \lambda_2 = 4\pi G g^2\varphi_0^4\left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right), \lambda_3 = \frac{g^2\varphi_0^2}{2\cos^2\theta}, \lambda_4 = 8\pi G, \quad (7.2.3)$$

we arrive at the following equivalent formulation of (7.2.2):

$$\Delta u = -\frac{\lambda_1}{4}e^{\upsilon+\eta} - \lambda_1 e^{u} + 4\pi\sum_{k=1}^{N}\delta(z - z_k) \quad (7.2.4)$$

$$\Delta(\eta + \lambda_4 e^{\upsilon}) = -\lambda_2 e^{\eta} + \lambda_3\lambda_4 e^{\eta+\upsilon} \quad (7.2.5)$$

$$\Delta\upsilon = \frac{\lambda_3}{\varphi_0^2}e^{\upsilon+\eta} - \lambda_3 e^{\eta} + \frac{\lambda_1}{2}e^{u}, \text{ in } \mathbb{R}^2. \quad (7.2.6)$$

Notice that the first equation (7.2.4) admits the stucture of a "singular" Liouville equation and thus suggests that we take the integrability property,

$$\int_{\mathbb{R}^2} e^{u} < +\infty, \quad (7.2.7)$$

as a "natural" boundary condition. Since (7.2.7) is scale invariant under the transformation

$$u(x) \longrightarrow u_{\varepsilon}(x) = u\left(\frac{x}{\varepsilon}\right) + 2\log\left(\frac{1}{\varepsilon}\right),$$

$\forall\varepsilon > 0$, we can consider the $\varepsilon-$scaled version of (7.2.4)–(7.2.6) by also transforming:

$$v(x) \longrightarrow v_\varepsilon(x) = v\left(\frac{x}{\varepsilon}\right) + 2\log\left(\frac{1}{\varepsilon}\right)$$
$$\eta(x) \longrightarrow \eta_\varepsilon(x) = \eta\left(\frac{x}{\varepsilon}\right) + 2\log\left(\frac{1}{\varepsilon}\right).$$

In terms of the unknowns $(u_\varepsilon, v_\varepsilon, \eta_\varepsilon)$, the system (7.2.4)–(7.2.6) takes the form:

$$\Delta u = -\varepsilon^2\frac{\lambda_1}{4}e^{v+\eta} - \lambda_1 e^u + 4\pi\sum_{k=1}^N \delta(z - \varepsilon z_k) \tag{7.2.8}$$
$$\Delta\left(\eta + \varepsilon^2\lambda_4 e^v\right) = -\lambda_2 e^\eta + \varepsilon^2\lambda_3\lambda_4 e^{\eta+v} \tag{7.2.9}$$
$$\Delta v = \frac{\varepsilon^2\lambda_3}{\varphi_0^2}e^{v+\eta} - \lambda_3 e^\eta + \frac{\lambda_1}{2}e^u \text{ in } \mathbb{R}^2. \tag{7.2.10}$$

Thus, we search for a solution of (7.2.8)–(7.2.10) "close" (in a suitable sense) to those of the system:

$$\Delta u^0 = -\lambda_1 e^{u^0} + 4\pi\sum_{k=1}^N \delta(z - \varepsilon z_k) \tag{7.2.11}$$
$$\Delta\eta^0 = -\lambda_2 e^{\eta^0} \tag{7.2.12}$$
$$\Delta v^0 = -\lambda_3 e^{\eta^0} + \frac{\lambda_1}{2}e^{u^0} \text{ in } \mathbb{R}^2, \tag{7.2.13}$$

for which we can exhibit an explicit solution. Indeed, as in Section 3.4, set

$$f(z) = (N+1)\prod_{k=1}^N (z - z_k), \quad F(z) = \int_0^z f(\xi)d\xi,$$

and for $\varepsilon > 0$, let

$$f_\varepsilon(z) = (N+1)\prod_{k=1}^N (z - \varepsilon z_k), \text{ and } F_\varepsilon(z) = \int_0^z f_\varepsilon(\xi)d\xi.$$

By (2.2.3) we see that the functions

$$u^0_{\varepsilon,a}(z) = \log\left[\frac{8|f_\varepsilon(z)|^2}{\lambda_1\left(1 + |F_\varepsilon(z) + a|^2\right)^2}\right], \quad \eta^0_b(z) = \log\left[\frac{8}{\lambda_2(1 + |z + b|^2)^2}\right]$$

satisfy (7.2.11) and (7.2.12) respectively, for every $\varepsilon > 0$ and $a, b \in \mathbb{C}$. Furthermore, if we set

$$\kappa = \frac{2\lambda_3}{\lambda_2}, \tag{7.2.14}$$

then we also solve (7.2.13) by taking

$$v^0_{\varepsilon,a,b} = \log\left[\frac{1+|F_\varepsilon(z)+a|^2}{(1+|z+b|^2)^\kappa}\right].$$

Reasonably, we may look for a solution of (7.2.4)–(7.2.6) in the form:

$$u(z) = u^0_{\varepsilon,a}(\varepsilon z) + 2\log\varepsilon + \varepsilon^2\sigma_1(\varepsilon z) \tag{7.2.15}$$

$$\eta(z) = \eta^0_b(\varepsilon z) + 2\log\varepsilon + \varepsilon^2\sigma_2(\varepsilon z) \tag{7.2.16}$$

$$v(z) = v^0_{\varepsilon,a,b}(\varepsilon z) + 2\log\varepsilon + \varepsilon^2\sigma_3(\varepsilon z) \tag{7.2.17}$$

with σ_1, σ_2, and σ_3 suitable functions which identify the error terms in the expansion (7.2.15)–(7.2.17), as $\varepsilon \to 0$. Introducing the notation:

$$u^0_{\varepsilon,a}(\varepsilon z) + 2\log\varepsilon := \log\rho^I_{\varepsilon,a}(z),$$
$$\eta^0_b(\varepsilon z) + 2\log\varepsilon := \log\rho^{II}_{\varepsilon,b}(z),$$
$$v^0_{\varepsilon,a,b}(\varepsilon z) + 2\log\varepsilon := \log\rho^{III}_{\varepsilon,a,b}(z),$$

we see that

$$\rho^I_{\varepsilon,a}(z) = \frac{8\varepsilon^{2N+2}|f(z)|^2}{\lambda_1\left(1+\varepsilon^{2N+2}\left|F(z)+\frac{a}{\varepsilon^{N+1}}\right|^2\right)^2},$$

$$\rho^{II}_{\varepsilon,b}(z) = \frac{8\varepsilon^2}{\lambda_2(1+|\varepsilon z+b|^2)^2},$$

$$\rho^{III}_{\varepsilon,a,b}(z) = \frac{\varepsilon^2\left(1+\varepsilon^{2N+2}\left|F(z)+\frac{a}{\varepsilon^{N+1}}\right|^2\right)}{(1+|\varepsilon z+b|^2)^\kappa},$$

which we may consider for negative ε as well. We prove:

Theorem 7.2.2 *Let $N \in \mathbb{N}$ be such that*

$$\kappa = \frac{2\lambda_3}{\lambda_2} > N+1. \tag{7.2.18}$$

For any given set of points $\{z_j\}_{j=1}^N \in \mathbb{R}^2$ (repeated according to their multiplicity), there exists $\varepsilon_1 > 0$, such that for every $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\varepsilon \neq 0$ and problem (7.2.4)–(7.2.6) *admits a solution $(u^\varepsilon, \eta^\varepsilon, v^\varepsilon)$ of the following form:*

$$u^\varepsilon(z) = \log\rho^I_{\varepsilon,a^*_\varepsilon}(z) + \varepsilon^2 w_1(\varepsilon|z|) + \varepsilon^2 u^*_{1,\varepsilon}(\varepsilon z), \tag{7.2.19}$$

$$\eta^\varepsilon(z) = \log\rho^{II}_{\varepsilon,b^*_\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon|z|) + \varepsilon^2 u^*_{2,\varepsilon}(\varepsilon z), \tag{7.2.20}$$

$$v^\varepsilon(z) = \log\rho^{III}_{\varepsilon,a^*_\varepsilon,b^*_\varepsilon}(z) + \varepsilon^2 w_3(\varepsilon|z|) + \varepsilon^2 u^*_{3,\varepsilon}(\varepsilon z), \tag{7.2.21}$$

with $\rho^{I}_{\varepsilon,a^*_\varepsilon}(z)$, $\rho^{II}_{\varepsilon,b^*_\varepsilon}(z)$, $\rho^{III}_{\varepsilon,a^*_\varepsilon,b^*_\varepsilon}(z)$ *defined as above and* $|a^*_\varepsilon| + |b^*_\varepsilon| \to 0$*, as* $\varepsilon \to 0$*. Furthermore, the functions* w_1, w_2*, and* w_3 *are radially symmetric and satisfy:*

$$w_1(|z|) = C_1 \log|z| + O(1) \tag{7.2.22}$$

$$w_2(|z|) = -C_2 \log|z| + O(1) \tag{7.2.23}$$

$$w_3(|z|) = C_3 \log|z| + O(1) \tag{7.2.24}$$

as $|z| \to \infty$*. The explicit constants* C_1, C_2*, and* C_3 *are given in Lemma 7.3.4 below; while* $u^*_{1,\varepsilon}$*,* $u^*_{2,\varepsilon}$*, and* $u^*_{3,\varepsilon}$ *satisfy*

$$\sup_{z\in\mathbb{R}^2} \frac{\sum_{j=1}^{3} |u^*_{j,\varepsilon}(\varepsilon z)|}{1+(\log|z|)^+} = o(1), \qquad \text{as } \varepsilon \to 0. \tag{7.2.25}$$

In particular,

$$e^{u_\varepsilon} \in \mathrm{L}^1(\mathbb{R}^2),\ e^{\eta_\varepsilon} \in \mathrm{L}^1(\mathbb{R}^2),\ \textit{and}\ |\nabla e^{v_\varepsilon}| \in \mathrm{L}^2(\mathbb{R}^2). \tag{7.2.26}$$

Remark 7.2.3 By our construction, the sufficient condition (7.2.18) is clearly also necessary to ensure the validity of the last boundary condition in (7.2.26). Notice that if the parameters λ_j, $j = 1, \cdots, 4$, are assigned by (7.2.3), then (7.2.18) reads as

$$\frac{\sin^2\theta}{4\pi G\varphi_0^2} > N+1,$$

and provides a sufficient condition for the existence of selfgravitating electroweak strings, as stated in Theorem 7.2.1. This condition is analogous to the necessary and sufficient condition obtained in [Y2] for the existence of abelian Higgs strings in the Einstein–Maxwell–Higgs system. It imposes a restriction between the total string number N and the gravitational constant G, which should be considered as a small parameter. Notice also that φ_0 in (7.2.1) plays the role of a symmetry-breaking parameter analogously to the abelian Higgs strings model.

Clearly Theorem 7.2.1 is a straight forward consequence of Theorem 7.2.2, and so we shall devote the next section to the proof of the latter.

7.3 The proof of Theorem 7.2.2

Following [ChI1], we derive our result by making an appropriate use of the Implicit Function theorem (see e.g., [Nir]) over the Hilbert spaces:

$$X_\alpha = \left\{ u \in \mathrm{L}^2_{loc}(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1+|x|^{2+\alpha})|u(x)|^2 dx < \infty \right\}$$

$$Y_\alpha = \left\{ u \in W^{2,2}_{loc}(\mathbb{R}^2) \mid \|\Delta u\|^2_{X_\alpha} + \left\| \frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}} \right\|^2_{L^2(\mathbb{R}^2)} < \infty \right\}$$

$\alpha \in (0, 1)$, already introduced in (3.4.11) with the relative norms. Since we are going to search for solutions (u, η, v) in the form (7.2.15)–(7.2.17), then by direct inspection we see that the functions σ_j , $j = 1, 2, 3$ must satisfy:

$$\Delta\sigma_1 = -\frac{\lambda_1}{4} g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(\sigma_2+\sigma_3)} - \frac{\lambda_1}{\varepsilon^2} g_{\varepsilon,a}^{I}(z)(e^{\varepsilon^2\sigma_1} - 1) \tag{7.3.1}$$

$$\Delta\sigma_2 = -\lambda_4 \Delta\left[g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2\sigma_3}\right] - \frac{\lambda_2}{\varepsilon^2} g_b^{II}(z)\left(e^{\varepsilon^2\sigma_2} - 1\right)$$
$$+\lambda_3\lambda_4 g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(\sigma_2+\sigma_3)} \tag{7.3.2}$$

$$\Delta\sigma_3 = \frac{\lambda_3}{\varphi_0^2} g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(\sigma_2+\sigma_3)} - \frac{\lambda_3}{\varepsilon^2} g_b^{II}(z)\left(e^{\varepsilon^2\sigma_2} - 1\right)$$
$$+\frac{\lambda_1}{2\varepsilon^2} g_{\varepsilon,a}^{I}(z)\left(e^{\varepsilon^2\sigma_1} - 1\right), \tag{7.3.3}$$

where we have set

$$g_{\varepsilon,a}^{I}(z) = e^{u_{\varepsilon,a}}, \quad g_b^{II}(z) = e^{\eta_b^0}, \quad g_{\varepsilon,a,b}^{III}(z) = e^{v_{\varepsilon,a,b}^0}.$$

To determine the triplet $(\sigma_1, \sigma_2, \sigma_3)$ we are going to consider the free parameters $a, b \in \mathbb{C}$ introduced above as part of our unknowns. We concentrate around the values $a = 0$ and $b = 0$, and define the radial functions:

$$\rho_1 = \lim_{\varepsilon\to 0} g_{\varepsilon,0}^{I} = \frac{8(N+1)^2 r^{2N}}{\lambda_1\left(1 + r^{2N+2}\right)^2}, \quad \rho_2 = g_0^{II} = \frac{8}{\lambda_2\left(1 + r^2\right)^2},$$

and

$$\rho_3 = \lim_{\varepsilon\to 0} g_{\varepsilon,0}^{III} = \frac{1 + r^{2N+2}}{(1 + r^2)^{\kappa}}.$$

Thus, by taking $a = b = 0$ in (7.3.1), (7.3.2), and (7.3.3) and by letting $\varepsilon \to 0$, (formally) we obtain the linear system:

$$\Delta w_1 + \lambda_1\rho_1 w_1 = -\frac{\lambda_1}{4}\rho_2\rho_3 \tag{7.3.4}$$
$$\Delta w_2 + \lambda_2\rho_2 w_2 = -\lambda_4\Delta\rho_3 + \lambda_3\lambda_4\rho_2\rho_3 \tag{7.3.5}$$
$$\Delta w_3 = \frac{1}{2}\lambda_1\rho_1 w_1 - \lambda_3\rho_2 w_2 + \frac{\lambda_3}{\varphi_0^2}\rho_2\rho_3. \tag{7.3.6}$$

Consequently, if (w_1, w_2, w_3) is a solution of (7.3.4), (7.3.5), (7.3.6) then, under the decomposition:

$$\sigma_j(z) = w_j(z) + u_j(z), \qquad j = 1, 2, 3, \tag{7.3.7}$$

we reduce to solve for (u_1, u_2, u_3) the following implicit problem:

$$
\begin{aligned}
P_1(u_1, u_2, u_3, a, b, \varepsilon) &= \Delta u_1 + \frac{\lambda_1}{4} g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(u_2+u_3+w_2+w_3)} \\
&\quad + \frac{\lambda_1}{\varepsilon^2} g_{\varepsilon,a}^{I}(z)(e^{\varepsilon^2(u_1+w_1)} - 1) + \Delta w_1 = 0, \\
P_2(u_1, u_2, u_3, a, b, \varepsilon) &= \Delta \left(u_2 + \lambda_4 g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(u_3+w_3)} \right) \\
&\quad + \frac{\lambda_2}{\varepsilon^2} g_b^{II}(z)(e^{\varepsilon^2(u_2+w_2)} - 1) \\
&\quad - \lambda_3 \lambda_4 g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(u_2+u_3+w_2+w_3)} + \Delta w_2 = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
P_3(u_1, u_2, u_3, a, b, \varepsilon) &= \Delta u_3 - \frac{\lambda_3}{\varphi_0^2} g_b^{II}(z) g_{\varepsilon,a,b}^{III}(z) e^{\varepsilon^2(u_2+u_3+w_2+w_3)} \\
&\quad + \frac{\lambda_3}{\varepsilon^2} g_b^{II}(z)(e^{\varepsilon^2(u_2+w_2)} - 1) \\
&\quad - \frac{\lambda_1}{2\varepsilon^2} g_{\varepsilon,a}^{I}(z)(e^{\varepsilon^2(u_1+w_1)} - 1) + \Delta w_3 = 0.
\end{aligned}
$$

Concerning the linear system (7.3.4)–(7.3.6), we have:

Lemma 7.3.4 *For $\kappa > N$, there exists a radial solution (w_1, w_2, w_3) of* (7.3.4)–(7.3.6) *in Y_α^3 satisfying*

$$w_1(r) = C_1 \log r + O(1), \quad \text{and} \quad \dot{w}_1(r) = \frac{C_1}{r} + O(1) \tag{7.3.8}$$

$$w_2(r) = -C_2 \log r + O(1), \quad \text{and} \quad \dot{w}_2(r) = -\frac{C_2}{r} + O(1) \tag{7.3.9}$$

$$w_3(r) = C_3 \log r + O(1), \quad \text{and} \quad \dot{w}_3(r) = \frac{C_3}{r} + O(1) \tag{7.3.10}$$

as $r \to \infty$, where:

$$
\begin{aligned}
C_1 &= \frac{\lambda_1}{\lambda_2} \left[\frac{\kappa(\kappa-1)\cdots(\kappa-N) - (N+1)!}{(1+\kappa)\kappa\cdots(\kappa-N)} \right], \ \textit{and so } C_1 > 0 \textit{ for } \kappa > N+1; \\
C_2 &= \frac{4(\lambda_2+\lambda_3)\lambda_4 \left[\kappa^2(\kappa-1)\cdots(\kappa-N) + (\kappa-2N-2)(N+1)! \right]}{\lambda_2(2+\kappa)(1+\kappa)\cdots(\kappa-N)}, \\
&\quad \textit{and so } C_2 > 0 \textit{ for } \kappa > N+1; \\
C_3 &= -\frac{C_1}{2} - C_2 \frac{\lambda_3}{\lambda_2} + \frac{4\mu}{(\kappa+1)\lambda_2};
\end{aligned}
$$

respectively, with $\mu = \frac{\lambda_3}{\varphi_0^2} - \frac{\lambda_3^2 \lambda_4}{\lambda_2} - \frac{\lambda_1}{8}$ and κ as defined in (7.2.14).

Before going into the proof of Lemma 7.3.4, we recall the following properties relative to the operators defined by the right-hand side of (7.3.4) and (7.3.5), as established in Proposition 3.4.19.

Proposition 7.3.5 *For $\alpha \in (0,1)$ and $j = 1, 2$, set*

$$L_j = \Delta + \lambda_j \rho_j : Y_\alpha \to X_\alpha.$$

We have

$$Ker L_j = Span\left\{\varphi_{j,+}, \varphi_{j,-}, \varphi_{j,0}\right\}, \tag{7.3.11}$$

where

$$\varphi_{1,+} = \frac{r^{N+1}\cos(N+1)\theta}{1+r^{2N+2}}, \qquad \varphi_{1,-} = \frac{r^{N+1}\sin(N+1)\theta}{1+r^{2N+2}},$$
$$\varphi_{2,+} = \frac{r\cos\theta}{1+r^2}, \qquad \varphi_{2,-} = \frac{r\sin\theta}{1+r^2},$$
$$\varphi_{1,0} = \frac{1-r^{2(N+1)}}{1+r^{2(N+1)}}, \qquad \varphi_{2,0} = \frac{1-r^2}{1+r^2}.$$

Moreover,

$$Im L_j = \left\{ f \in X_\alpha \,|\, \int_{\mathbb{R}^2} f\varphi_{j,\pm} = 0 \right\}. \tag{7.3.12}$$

Proof of Lemma 7.3.4. Taking into account (3.4.26), we know that a radial solution in Y_α^r of the equation

$$\Delta w(r) + \lambda_1 \rho_1 w(r) = f(r), \tag{7.3.13}$$

is given by the formula:

$$\begin{aligned} w(r) = &\left(\varphi_{1,0}(r)\log r + \frac{2}{(N+1)(1+r^{2(N+1)})}\right)\int_0^r \varphi_{1,0}(t) f(t) t\,dt \\ &-\varphi_{1,0}(r)\int_0^r \left(\varphi_{1,0}(t)\log t + \frac{2}{(N+1)(1+t^{2(N+1)})}\right) f(t) t\,dt. \end{aligned} \tag{7.3.14}$$

Furthermore, setting $c_f = \int_0^{+\infty} \varphi_{1,0}(t) f(t) t\,dt$, then from Corollary 3.4.21, we also know that

$$w(r) = -c_f \log r + O(1),$$
$$\dot{w}(r) = -\frac{c_f}{r} + O(1),$$

as $r \to +\infty$. To obtain w_1, we use formula (7.3.14) with $f(r) = -\frac{\lambda_1}{4}\rho_2(r)\rho_3(r)$. So we can check (7.3.8) with $C_1 = \frac{\lambda_1}{4} A_1$ where

$$\begin{aligned} A_1 = A_1(\infty) &= \int_0^\infty \varphi_{1,0}(r) r \rho_2(r)\rho_3(r) dr = \frac{8}{\lambda_2}\int_0^\infty \frac{(1-r^{2N+2})r}{(1+r^2)^{2+\kappa}} dr \\ &= \frac{4}{\lambda_2}\int_0^\infty \frac{1-t^{N+1}}{(1+t)^{2+\kappa}} dt = \frac{4}{\lambda_2}\left[\frac{1}{1+\kappa} - \frac{(N+1)!}{(1+\kappa)\kappa\cdots(\kappa-N)}\right] \\ &= \frac{4}{\lambda_2}\left[\frac{\kappa(\kappa-1)\cdots(\kappa-N) - (N+1)!}{(1+\kappa)\kappa\cdots(\kappa-N)}\right]. \end{aligned}$$

So, $A_1 > 0$ for $\kappa > N+1$ and (7.3.8) is proved.

To obtain w_2, we use the analogous form of formula (7.3.14) for the operator L_2, however now we take $N = 0$ and $\varphi_{2,0}$ in place of $\varphi_{1,0}$. Exactly as above, we reduce to evaluate

$$C_2 = \int_0^\infty \varphi_{2,0}(r) f(r) r dr, \tag{7.3.15}$$

with $f(r) = \lambda_3\lambda_4\rho_2\rho_3 - \lambda_4\Delta\rho_3$. Since $\varphi_{2,0} \in KerL_2$, integration by parts leads to the identity,

$$\int_0^\infty \varphi_{2,0}\Delta\rho_3 r dr = \int_0^\infty \Delta\varphi_{2,0}\rho_3 r dr = -\lambda_2 \int_0^\infty \varphi_{2,0}\rho_2\rho_3 r dr. \tag{7.3.16}$$

Consequently,

$$\begin{aligned}
A_2 &= (\lambda_2+\lambda_3)\lambda_4 \int_0^\infty \varphi_{2,0}\rho_2\rho_3 r dr \\
&= \frac{8(\lambda_2+\lambda_3)\lambda_4}{\lambda_2} \int_0^\infty \frac{(1-r^2)(1+r^{2N+2})}{(1+r^2)^{3+\kappa}} r dr \\
&= \frac{4(\lambda_2+\lambda_3)\lambda_4}{\lambda_2} \int_0^\infty \frac{(1-t)\left(1+t^{N+1}\right)}{(1+t)^{3+\kappa}} dt \\
&= \frac{4(\lambda_2+\lambda_3)\lambda_4}{\lambda_2} \int_0^\infty \left[\frac{1}{(1+t)^{3+\kappa}} - \frac{t}{(1+t)^{3+\kappa}} + \frac{t^{N+1}}{(1+t)^{3+\kappa}} - \frac{t^{N+2}}{(1+t)^{3+\kappa}}\right] dt \\
&= \frac{4(\lambda_2+\lambda_3)\lambda_4}{\lambda_2} \left[\frac{1}{2+\kappa} - \frac{1}{(2+\kappa)(1+\kappa)} + \frac{(N+1)!}{(2+\kappa)(1+\kappa)\cdots(1+\kappa-N)}\right. \\
&\qquad\qquad \left. - \frac{(N+2)!}{(2+\kappa)(1+\kappa)\cdots(\kappa-N)}\right] \\
&= \frac{4(\lambda_2+\lambda_3)\lambda_4}{\lambda_2(2+\kappa)(1+\kappa)\cdots(\kappa-N)} [(\kappa+1)\kappa\cdots(\kappa-N) - \kappa(\kappa-1)\cdots(\kappa-N) \\
&\qquad\qquad + (\kappa-N)(N+1)! - (N+2)!] \\
&= \frac{4(\lambda_2+\lambda_3)\lambda_4[\kappa^2(\kappa-1)\cdots(\kappa-N) + (\kappa-2N-2)(N+1)!]}{\lambda_2(2+\kappa)(1+\kappa)\cdots(\kappa-N)},
\end{aligned} \tag{7.3.17}$$

and (7.3.9) is also proved.

In order to obtain w_3 (with the given asymptotic expansion), we use the decomposition

$$w_3(r) = -\frac{w_1(r)}{2} + \frac{\lambda_3}{\lambda_2} w_2(r) + \frac{\lambda_3\lambda_4}{\lambda_2}\rho_3(r) + \varphi(r), \tag{7.3.18}$$

where φ is a regular radial function satisfying:

$$\Delta\varphi = \left(\frac{\lambda_3}{\varphi_0^2} - \frac{\lambda_3^2\lambda_4}{\lambda_2} - \frac{\lambda_1}{8}\right)\rho_2\rho_3.$$

Set

$$\mu = \frac{\lambda_3}{\varphi_0^2} - \frac{\lambda_3^2 \lambda_4}{\lambda_2} - \frac{\lambda_1}{8}. \tag{7.3.19}$$

Incidentally, notice that the choice of λ_j, $j = 1, \cdots, 4$, in (7.2.3) gives $\mu = \frac{g_1^2}{2} \sin^4 \theta (1 + \cos^2 \theta)$. We have

$$\begin{aligned} r\dot{\varphi}(r) &= \frac{8\mu}{\lambda_2} \int_0^r \frac{(1 + r^{2N+2})r}{(1 + r^2)^{\kappa+2}} dr = \frac{4\mu}{\lambda_2} \int_0^{r^2} \frac{1 + t^{N+1}}{(1 + t)^{\kappa+2}} dt \\ &= \frac{4\mu}{\lambda_2(\kappa + 1)} \left(1 - \frac{1}{(1 + r^2)^{\kappa+1}}\right) + \frac{4\mu}{\lambda_2} \int_0^{r^2} \frac{t^{N+1}}{(1 + t)^{\kappa+2}} dt. \end{aligned}$$

Since $\kappa > N$, $r\dot{\varphi}(r) \to \frac{4\mu}{\lambda_2(\kappa+1)}$, as $r \to +\infty$ and thus,

$$\varphi(r) = \frac{4\mu}{(\kappa + 1)\lambda_2} \log r + O(1).$$

In view of (7.3.18), we derive the desired conclusion for w_3, and we complete the proof. □

Remark 7.3.6 With the choice of (w_1, w_2, w_3) in Lemma 7.3.4 and for the condition $\kappa > N + 1$, we see that for $0 < \alpha < \min\{\frac{1}{2}, \kappa - N - 1\}$ there exists $\varepsilon_0 > 0$ such that the operator $P = (P_1, P_2, P_3)$ defined above is a continuous mapping from $\Omega_{\varepsilon_0} = \{(u, a, b, \varepsilon) \in {Y_\alpha}^3 \times \mathbb{C}^2 \times \mathbb{R} : \|u\|_{{Y_\alpha}^3} + |a| + |b| + |\varepsilon| < \varepsilon_0\}$ into ${X_\alpha}^3$ and $P(0, 0, 0, 0, 0, 0) = 0$.

Next we proceed to compute the linearized operator of P around zero. From tedious but not difficult computations we see that, for $a = a_1 + ia_2$ and $b = b_1 + ib_2$, we have

$$\left.\frac{\partial g^I_{\varepsilon,a}(z)}{\partial a_1}\right|_{(a,\varepsilon)=(0,0)} = -4\rho_1\varphi_{1,+}, \qquad \left.\frac{\partial g^I_{\varepsilon,a}(z)}{\partial a_2}\right|_{(a,\varepsilon)=(0,0)} = -4\rho_1\varphi_{1,-},$$

$$\left.\frac{\partial g^{II}_b(z)}{\partial b_1}\right|_{b=0} = -4\rho_2\varphi_{2,+}, \qquad \left.\frac{\partial g^{II}_b(z)}{\partial b_2}\right|_{b=0} = -4\rho_2\varphi_{2,-},$$

$$\left.\frac{\partial g^{III}_{\varepsilon,a,b}(z)}{\partial a_1}\right|_{(a,b,\varepsilon)=(0,0,0)} = 2\rho_3\varphi_{1,+}, \qquad \left.\frac{\partial g^{III}_{\varepsilon,a,b}(z)}{\partial a_2}\right|_{(a,b,\varepsilon)=(0,0,0)} = 2\rho_3\varphi_{1,-},$$

$$\left.\frac{\partial g^{III}_{\varepsilon,a,b}(z)}{\partial b_1}\right|_{(a,b,\varepsilon)=(0,0,0)} = -\frac{4\lambda_3}{\lambda_2}\rho_3\varphi_{2,+}, \qquad \left.\frac{\partial g^{III}_{\varepsilon,a,b}(z)}{\partial b_2}\right|_{(a,b,\varepsilon)=(0,0,0)} = -\frac{4\lambda_3}{\lambda_2}\rho_3\varphi_{2,-},$$

$$\left.\frac{\partial g^{II}_b(z) g^{III}_{\varepsilon,a,b}(z)}{\partial a_1}\right|_{(a,b,\varepsilon)=(0,0,0)} = 2\rho_2\rho_3\varphi_{1,+}, \qquad \left.\frac{\partial g^{II}_b(z) g^{III}_{\varepsilon,a,b}(z)}{\partial a_2}\right|_{(a,b,\varepsilon)=(0,0,0)} = 2\rho_2\rho_3\varphi_{1,-},$$

$$\left.\frac{\partial g^{II}_b(z) g^{III}_{\varepsilon,a,b}(z)}{\partial b_1}\right|_{(a,b,\varepsilon)=(0,0,0)} = -4\left(1 + \frac{\lambda_3}{\lambda_2}\right)\rho_2\rho_3\varphi_{2,+},$$

$$\left.\frac{\partial g^{II}_b(z) g^{III}_{\varepsilon,a,b}(z)}{\partial b_2}\right|_{(a,b,\varepsilon)=(0,0,0)} = -4\left(1 + \frac{\lambda_3}{\lambda_2}\right)\rho_2\rho_3\varphi_{2,-}.$$

Therefore setting

$$P'_{(u_1,u_2,u_3,a,b)}(0,0,0,0,0,0)[v_1,v_2,v_3,\alpha,\beta] = \mathcal{A}[v_1,v_2,v_3,\alpha,\beta],$$

we can check that for $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, we have:

$$\begin{aligned}\mathcal{A}_1[v_1,v_2,v_3,\alpha,\beta] = {} & \Delta v_1 + \lambda_1\rho_1 v_1 + \lambda_1\left[-4\rho_1 w_1 + \frac{1}{2}\rho_2\rho_3\right](\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2) \\ & -\lambda_1\left(\frac{\lambda_3}{\lambda_2}+1\right)\rho_2\rho_3(\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2),\end{aligned} \tag{7.3.20}$$

$$\begin{aligned}\mathcal{A}_2[v_1,v_2,v_3,\alpha,\beta] = {} & \Delta v_2 + \lambda_2\rho_2 v_2 - 2\lambda_3\lambda_4\rho_2\rho_3(\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2) \\ & -2\lambda_4\Delta\left[\rho_3(\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2)\right] \\ & -4\frac{\lambda_4\lambda_3}{\lambda_2}\Delta\left[\rho_3(\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2)\right] \\ & -4\left[\lambda_2\rho_2 w_2 - \lambda_3\lambda_4\left(1+\frac{\lambda_3}{\lambda_2}\right)\rho_2\rho_3\right](\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2),\end{aligned} \tag{7.3.21}$$

and

$$\begin{aligned}\mathcal{A}_3[v_1,v_2,v_3,\alpha,\beta] = {} & \Delta v_3 + \lambda_3\rho_2 v_2 - \tfrac{\lambda_1}{2}\rho_1 v_1 \\ & +\left[2\lambda_1\rho_1 w_1 - \tfrac{2\lambda_3}{\phi_0^2}\rho_2\rho_3\right](\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2) \\ & -\left[4\lambda_3\rho_2 w_1 - \tfrac{4\lambda_3}{\phi_0^2}\left(\tfrac{\lambda_3}{\lambda_2}+1\right)\rho_2\rho_3\right]\left(\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2\right).\end{aligned} \tag{7.3.22}$$

It is interesting to note that although we need the condition $\kappa > N+1$ to ensure that the operator P is well defined from ${Y_\alpha}^3 \times \mathbb{C}^2 \times (-\varepsilon_0, \varepsilon_0)$ into ${X_\alpha}^3$, its linearized operator at the origin $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ (given in (7.3.20)–(7.3.22)) only appears to be well defined from ${Y_\alpha}^3 \times \mathbb{C}^2$ into ${X_\alpha}^3$ under the weaker assumption $\kappa > N$, which also suffices to ensure the following crucial properties:

Proposition 7.3.7 *If $\kappa > N$, then the operator $\mathcal{A} : (Y_\alpha)^3 \times (\mathbb{C})^2 \to (X_\alpha)^3$ given by (7.3.20)–(7.3.22) is onto. Moreover,*

$$\begin{aligned}Ker\mathcal{A} = Span \Big\{ & (0,0,1); \left(\varphi_{1,\pm}, \varphi_{2,\pm}, -\tfrac{1}{2}\varphi_{1,\pm} + \tfrac{\lambda_3}{\lambda_2}\varphi_{2,\pm}\right); \left(\varphi_{1,0}, \varphi_{2,0}, -\tfrac{1}{2}\varphi_{1,0} + \tfrac{\lambda_3}{\lambda_2}\varphi_{2,0}\right); \\ & \left(\varphi_{1,\pm}, \varphi_{2,0}, -\tfrac{1}{2}\varphi_{1,\pm} + \tfrac{\lambda_3}{\lambda_2}\varphi_{2,0}\right); \left(\varphi_{1,0}, \varphi_{2,\pm}, -\tfrac{1}{2}\varphi_{1,0} + \tfrac{\lambda_3}{\lambda_2}\varphi_{2,\pm}\right)\Big\} \times \{(0,0)\}.\end{aligned} \tag{7.3.23}$$

In order to prove the statement above, we establish the following:

Lemma 7.3.8 *Let $\kappa > N$. Then*

$$I_1^\pm := \int_{\mathbb{R}^2}\left[-4\rho_1 w_1 + \frac{1}{2}\rho_2\rho_3\right]\varphi_{1,\pm}^2 dx = \frac{2\pi}{\lambda_2(\kappa+1)}, \tag{7.3.24}$$

and

$$
\begin{aligned}
I_2^{\pm} &:= \int_{\mathbb{R}^2} \left[-\lambda_2 \rho_2 w_2 + \lambda_3 \lambda_4 \left(\frac{\lambda_3}{\lambda_2} + 1 \right) \rho_2 \rho_3 \right] \varphi_{2,\pm}^2 dx \\
&\quad - \frac{\lambda_3 \lambda_4}{\lambda_2} \int_{\mathbb{R}^2} \Delta(\rho_3 \varphi_{2,\pm}) \varphi_{2,\pm} dx \\
&= \frac{\pi \lambda_4 (N+1)!(N+1)}{(1+\kappa)\kappa \cdots (1+\kappa - N)},
\end{aligned}
\tag{7.3.25}
$$

with w_1 and w_2 as given in Lemma 7.3.4.

Proof. We prove (7.3.24) by recalling the formula

$$
L_1 \left[\frac{1}{(1+r^{2N+2})^2} \right] = \frac{16(N+1)^2 r^{4N+2}}{(1+r^{2N+2})^4},
$$

and computing

$$
\begin{aligned}
I_1^{\pm} &= \int_0^{2\pi} \int_0^{\infty} \left[-4\rho_1 w_1 + \frac{1}{2} \rho_2 \rho_3 \right] \frac{r^{2N+2}}{(1+r^{2N+2})^2} \begin{Bmatrix} \cos^2(N+1)\theta \\ \sin^2(N+1)\theta \end{Bmatrix} r dr d\theta \\
&= \pi \int_0^{\infty} \left[-\frac{32(N+1)^2 r^{2N}}{\lambda_1 (1+r^{2N+2})^2} w_1 + \frac{1}{2} \rho_2 \rho_3 \right] \frac{r^{2N+2}}{(1+r^{2N+2})^2} r dr \\
&= \pi \int_0^{\infty} \left\{ -\frac{2}{\lambda_1} L_1 \left[\frac{1}{(1+r^{2N+2})^2} \right] w_1 + \frac{\rho_2 \rho_3 r^{2N+2}}{2(1+r^{2N+2})^2} \right\} r dr \\
&= \pi \int_0^{\infty} \left\{ -\frac{2}{\lambda_1} \frac{L_1 w_1}{(1+r^{2N+2})^2} + \frac{\rho_2 \rho_3 r^{2N+2}}{2(1+r^{2N+2})^2} \right\} r dr \\
&= \pi \int_0^{\infty} \left\{ \frac{\rho_2 \rho_3}{2(1+r^{2N+2})^2} + \frac{\rho_2 \rho_3 r^{2N+2}}{2(1+r^{2N+2})^2} \right\} r dr \\
&= \frac{\pi}{2} \int_0^{\infty} \frac{\rho_2 \rho_3}{(1+r^{2N+2})} r dr = \frac{4\pi}{\lambda_2} \int_0^{\infty} \frac{r dr}{(1+r^2)^{\kappa+2}} = \frac{2\pi}{\lambda_2(\kappa+1)},
\end{aligned}
$$

where the integration by parts performed above is justified by virtue of the asymptotic behavior of w_1 and its derivative as $r \to +\infty$, as provided in Lemma 7.3.4. In order to prove (7.3.25), again we use integration by parts to obtain:

$$
\begin{aligned}
I_2^{\pm} &= \int_{\mathbb{R}^2} \left[-\lambda_2 \rho_2 w_2 + \lambda_3 \lambda_4 \left(1 + \frac{\lambda_3}{\lambda_2} \right) \rho_2 \rho_3 \right] \varphi_{2,\pm}^2 dx - \frac{\lambda_3 \lambda_4}{\lambda_2} \int_{\mathbb{R}^2} \rho_3 \varphi_{2,\pm} \Delta \varphi_{2,\pm} dx \\
&= \int_{\mathbb{R}^2} \left[-\lambda_2 \rho_2 w_2 + \lambda_3 \lambda_4 \left(2 + \frac{\lambda_3}{\lambda_2} \right) \rho_2 \rho_3 \right] \varphi_{2,\pm}^2 dx,
\end{aligned}
\tag{7.3.26}
$$

where we used $-\Delta \varphi_{2,\pm} = \lambda_2 \rho_2 \varphi_{2,\pm}$. In view of the identity

$$
L_2 \left[\frac{1}{(1+r^2)^2} \right] = \frac{16 r^2}{(1+r^2)^4},
$$

we may transform the first term of $I_2^{\pm}$ as follows

$$\int_{\mathbb{R}^2} \lambda_2\rho_2 w_2 \varphi_{2,\pm}^2 dx = -\int_0^\infty \int_0^{2\pi} \lambda_2\rho_2 w_2 \frac{r^2}{(1+r^2)^2} \begin{Bmatrix} \cos^2\theta \\ \sin^2\theta \end{Bmatrix} r dr d\theta$$
$$= -8\pi \int_0^\infty \frac{r^2}{(1+r^2)^4} w_2 r dr = -\frac{\pi}{2}\int_0^\infty L_2\left[\frac{1}{(1+r^2)^2}\right] w_2 r dr$$
$$= -\frac{\pi}{2}\int_0^\infty \frac{L_2 w_2}{(1+r^2)^2} r dr$$
$$= -\frac{\pi}{2}\int_0^\infty \frac{1}{(1+r^2)^2}\left[\lambda_3\lambda_4\rho_2\rho_3 - \lambda_4\Delta\rho_3\right] r dr,$$

where we used (7.3.9) to derive the last identity. Substituting this result into (7.3.26), we find

$$I_2^{\pm} = -\frac{\pi}{2}\lambda_3\lambda_4\int_0^\infty \frac{\rho_2\rho_3}{(1+r^2)^2} r dr + \frac{\pi}{2}\lambda_4\int_0^\infty \frac{\Delta\rho_3}{(1+r^2)^2} r dr$$
$$+\pi\lambda_3\lambda_4\left(2+\frac{\lambda_3}{\lambda_2}\right)\int_0^\infty \frac{\rho_2\rho_3 r^3}{(1+r^2)^2} dr = J_1 + J_2 + J_3.$$

We can rewrite J_1 and J_3 as follows:

$$J_1 = -\frac{\pi}{16}\lambda_2\lambda_3\lambda_4\int_0^\infty \rho_2^2\rho_3 r dr, \tag{7.3.27}$$
$$J_3 = \frac{\pi}{8}\lambda_2\lambda_3\lambda_4\left(2+\frac{\lambda_3}{\lambda_2}\right)\int_0^\infty \rho_2^2\rho_3 r^3 dr. \tag{7.3.28}$$

Also we can easily check that

$$\Delta\rho_2 = \lambda_2(2r^2-1)\rho_2^2.$$

Therefore, for $\kappa > N$ we can perform integration by parts and obtain

$$J_2 = \frac{\pi}{16}\lambda_2\lambda_4\int_0^\infty \Delta\rho_3\rho_2 r dr = \frac{\pi}{16}\lambda_2\lambda_4\int_0^\infty \rho_3\Delta\rho_2 r dr$$
$$= \frac{\pi}{16}\lambda_2^2\lambda_4\int_0^\infty (2r^3-r)\rho_2^2\rho_3 dr. \tag{7.3.29}$$

Consequently,

$$I_2^{\pm} = J_1+J_2+J_3 = \frac{\pi}{16}\lambda_2\lambda_3\lambda_4\int_0^\infty \left[\left(4+2\frac{\lambda_3}{\lambda_2}\right)r^3 - r\right]\rho_2^2\rho_3 dr$$
$$+\frac{\pi}{16}\lambda_2^2\lambda_4\int_0^\infty (2r^3-r)\rho_2^2\rho_3 dr$$
$$= \frac{\pi}{32}\lambda_2^2\lambda_4\kappa\int_0^\infty \left[(4+\kappa)r^3 - r\right]\rho_2^2\rho_3 dr \tag{7.3.30}$$
$$+\frac{\pi}{16}\lambda_2^2\lambda_4\int_0^\infty (2r^3-r)\rho_2^2\rho_3 dr$$
$$= \frac{\pi}{32}\lambda_2^2\lambda_4(\kappa+2)\left[(\kappa+2)K_1 - K_2\right],$$

where

$$K_1 = \int_0^\infty r^3 \rho_2^2 \rho_3 dr \quad \text{and} \quad K_2 = \int_0^\infty r \rho_2^2 \rho_3 dr.$$

We evaluate

$$\begin{aligned} K_1 &= \frac{64}{\lambda_2^2} \int_0^\infty \frac{r^3(1+r^{2N+2})}{(1+r^2)^{4+\kappa}} dr \\ &= \frac{32}{\lambda_2^2} \left[\int_0^\infty \frac{t}{(1+t)^{4+\kappa}} dt + \int_0^\infty \frac{t^{N+2}}{(1+t)^{4+\kappa}} dt \right] \\ &= \frac{32}{\lambda_2^2} \left[\frac{1}{(3+\kappa)(2+\kappa)} + \frac{(N+2)!}{(3+\kappa)(2+\kappa)\cdots(1+\kappa-N)} \right], \end{aligned} \tag{7.3.31}$$

and

$$\begin{aligned} K_2 &= \frac{64}{\lambda_2^2} \int_0^\infty \frac{r(1+r^{2N+2})}{(1+r^2)^{4+\kappa}} dr \\ &= \frac{32}{\lambda_2^2} \left[\int_0^\infty \frac{1}{(1+t)^{4+\kappa}} dt + \int_0^\infty \frac{t^{N+1}}{(1+t)^{4+\kappa}} dt \right] \\ &= \frac{32}{\lambda_2^2} \left[\frac{1}{3+\kappa} + \frac{(N+1)!}{(3+\kappa)(2+\kappa)\cdots(2+\kappa-N)} \right]. \end{aligned} \tag{7.3.32}$$

Substituting (7.3.31) and (7.3.32) into (7.3.30), we obtain

$$\begin{aligned} I_2^{\pm} &= \pi(\kappa+2)\lambda_4 \left[\frac{1}{3+\kappa} + \frac{(N+2)!}{(3+\kappa)(1+\kappa)\kappa\cdots(1+\kappa-N)} - \frac{1}{3+\kappa} \right. \\ &\qquad \left. - \frac{(N+1)!}{(3+\kappa)(2+\kappa)\cdots(2+\kappa-N)} \right] \\ &= \frac{\pi(\kappa+2)\lambda_4(N+1)![(N+2)(2+\kappa)-(1+\kappa-N)]}{(3+\kappa)(2+\kappa)\cdots(1+\kappa-N)} \\ &= \frac{\pi\lambda_4(N+1)!(N+1)}{(1+\kappa)\kappa\cdots(1+\kappa-N)}. \end{aligned}$$

The proof of Lemma 7.3.8 is completed. □

Proof of Proposition 7.3.7. Given $f = (f_1, f_2, f_3) \in (X_\alpha)^3$, we need to show the solvability in ${Y_\alpha}^3 \times \mathbb{C}^2$ of the linear equation:

$$A[v_1, v_2, v_3, \alpha, \beta] = f. \tag{7.3.33}$$

Equivalently,

$$\begin{aligned} &L_1 v_1 + \lambda_1 \left[-4\rho_1 w_1 + \frac{1}{2}\rho_2\rho_3 \right] (\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2) \\ &\quad -\lambda_1 \left(\frac{\lambda_3}{\lambda_2} + 1 \right) \rho_2\rho_3(\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2) = f_1, \end{aligned} \tag{7.3.34}$$

$$\begin{aligned} &L_2 v_2 - 2\lambda_3\lambda_4\rho_2\rho_3(\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2) - 2\lambda_4\Delta[\rho_3(\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2)] \\ &-4\left[\lambda_2\rho_2 w_2 - \lambda_3\lambda_4\left(\frac{\lambda_3}{\lambda_2}+1\right)\rho_2\rho_3\right](\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2) \\ &-4\frac{\lambda_4\lambda_3}{\lambda_2}\Delta[\rho_3(\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2)] = f_2, \end{aligned} \tag{7.3.35}$$

$$\begin{aligned} &\Delta v_3 + \lambda_3\rho_2 v_2 - \frac{\lambda_1}{2}\rho_1 v_1 + \left[2\lambda_1\rho_1 w_1 - \frac{2\lambda_3}{\varphi_0^2}\rho_2\rho_3\right](\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2) \\ &-\left[4\lambda_3\rho_2 w_1 - \frac{4\lambda_3}{\varphi_0^2}\left(\frac{\lambda_3}{\lambda_2}+1\right)\rho_2\rho_3\right](\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2) = f_3. \end{aligned} \tag{7.3.36}$$

By the orthogonality property of the system $\{\varphi_{1,\pm}, \varphi_{2,\pm}\}$ and by (7.3.24), we can explicitly determine

$$\alpha_1 = -\frac{\lambda_2(\kappa+1)}{2\pi\lambda_1}\int_{\mathbb{R}^2} f_1\varphi_{1,+}, \quad \alpha_2 = -\frac{\lambda_2(\kappa+1)}{2\pi\lambda_1}\int_{\mathbb{R}^2} f_1\varphi_{1,-}$$

in (7.3.34), and thus ensure that

$$(L_1 v_1, \varphi_{1,\pm})_{L^2} = 0. \tag{7.3.37}$$

Similarly by (7.3.25), we can determine β_1, β_2 in (7.3.35) such that,

$$(L_2 v_2, \varphi_{2,\pm})_{L^2} = 0. \tag{7.3.38}$$

With such choices for α_1, α_2 and β_1, β_2 we are in a position to use Proposition 7.3.7, to obtain the $v_1, v_2 \in Y_\alpha$, solution, respectively, to (7.3.34) and (7.3.35). At this point, set

$$\begin{aligned} g = &-\lambda_3\rho_2 v_2 + \frac{\lambda_1}{2}\rho_1 v_1 - \left[2\lambda_1\rho_1 w_1 - \frac{2\lambda_3}{\varphi_0^2}\rho_2\rho_3\right](\varphi_{1,+}\alpha_1 + \varphi_{1,-}\alpha_2) \\ &+\left[4\lambda_3\rho_2 w_1 - \frac{4\lambda_3}{\varphi_0^2}\left(\frac{\lambda_3}{\lambda_2}+1\right)\rho_2\rho_3\right](\varphi_{2,+}\beta_1 + \varphi_{2,-}\beta_2) + f_3 \in X_\alpha, \end{aligned} \tag{7.3.39}$$

and observe that (7.3.36) is solvable in Y_α with the corresponding solution given by

$$v_3(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|)g(y)dy + C \tag{7.3.40}$$

for a constant $C \in \mathbb{R}$. So the operator $\mathcal{A}$ is onto. Furthermore, $Ker\mathcal{A}$ can be determined by letting $f_1 = f_2 = f_3 = 0$ in the above argument. This leads to $\alpha_1 = 0 = \alpha_2$, $\beta_1 = 0 = \beta_2$ and $v_3 = -\frac{1}{2}v_1 + \frac{\lambda_3}{\lambda_2}v_2 + C$ with $v_j \in KerL_j$, $j = 1, 2$, and any constant $C \in \mathbb{R}$. Therefore, the desired conclusion (7.3.23) follows once we take into account Proposition 7.3.7. □

Proof of Theorem 7.2.2. We decompose $(Y_\alpha)^3 \times \mathbb{C}^2 = U_\alpha \oplus Ker\mathcal{A}$ with $U_\alpha = (Ker\mathcal{A})^\perp$, so that

$$\mathcal{A} = P'_{(u_1,u_2,u_3,a,b)}(0,0,0,0,0,0) : U_\alpha \to (X_\alpha)^3$$

defines an isomorphism. The standard Implicit Function theorem (see e.g., [Nir]) applies to the operator $P : U_\alpha \times (-\varepsilon_0, \varepsilon_0) \to (X_\alpha)^3$ for sufficiently small $\varepsilon_0 > 0$, and implies that there exist $\varepsilon_1 \in (0, \varepsilon_0)$ and a continuous function,

$$\varepsilon \mapsto \psi_\varepsilon = (u^*_{1,\varepsilon}, u^*_{2,\varepsilon}, u^*_{3,\varepsilon}, a^*_\varepsilon, b^*_\varepsilon),$$

from $(-\varepsilon_1, \varepsilon_1)$ into a neighborhood of the origin in U_α such that

$$P(u^*_{1,\varepsilon}, u^*_{2,\varepsilon}, u^*_{3,\varepsilon}, a^*_\varepsilon, b^*_\varepsilon, \varepsilon) = 0, \quad \text{for all } \varepsilon \in (-\varepsilon_1, \varepsilon_1),$$

and $u^*_{j,\varepsilon=0} = 0$ for every $j = 1,2,3$, and $a^*_{\varepsilon=0} = 0 = b^*_{\varepsilon=0}$. Consequently,

$$\begin{aligned} u^\varepsilon(z) &= \log \rho^I_{\varepsilon,a^*_\varepsilon}(z) + \varepsilon^2 w_1(\varepsilon z) + \varepsilon^2 u^*_{1,\varepsilon}(\varepsilon z), \\ \eta^\varepsilon(z) &= \log \rho^{II}_{\varepsilon,b^*_\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon z) + \varepsilon^2 u^*_{2,\varepsilon}(\varepsilon z), \\ v^\varepsilon(z) &= \log \rho^{III}_{\varepsilon,a^*_\varepsilon,b^*_\varepsilon}(z) + \varepsilon^2 w_3(\varepsilon z) + \varepsilon^2 u^*_{3,\varepsilon}(\varepsilon z), \end{aligned} \tag{7.3.41}$$

defines a solution for the system (7.2.4)–(7.2.6), $\forall \varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\varepsilon \neq 0$. Furthermore, from Lemma 3.4.18 we have

$$|u^*_{j,\varepsilon}(x)| \le C\|u^*_{j,\varepsilon}\|_{Y_\alpha}(\log^+|x| + 1) \le C\|\psi_\varepsilon\|_{U_\alpha}(\log^+|x| + 1), \quad j = 1,2,3,$$

with

$$\|\psi_\varepsilon\|_{U_\alpha} \to 0, \qquad \text{as } \varepsilon \to 0.$$

Therefore,

$$\sup_{\mathbb{R}^2} \frac{|u^*_{j,\varepsilon}(\varepsilon x)|}{\left(1 + \log^+|x|\right)} = o(1) \tag{7.3.42}$$

as $\varepsilon \to 0$. Since (7.2.18) holds, the explicit form of $\rho^I_{\varepsilon,a^*_\varepsilon}(z)$, $\rho^{II}_{\varepsilon,b^*_\varepsilon}(z)$, $\rho^{III}_{\varepsilon,a^*_\varepsilon,b^*_\varepsilon}(z)$, together with the asymptotic behaviors of w_1, w_2, w_2, as described in Lemma 7.3.4 and (7.3.42), imply that the solution $(u^\varepsilon, \eta^\varepsilon, v^\varepsilon)$ in (7.3.41) also satisfies the integral condition (7.2.26). The proof is completed. □

Final Remarks: By a more complete application of the Implicit Function theorem (e.g., [Nir]), we can actually claim the existence of a family of solutions that depend up on a number of parameters being equal to the dimension of $Ker\mathcal{A}$. A minor modification of the proof presented above allows us to include an equality in (7.2.18). In this case, the image of the operator P is mapped into the space $(X_{\alpha-\delta_0})^3$ for $\delta_0 > 0$ sufficiently small. Notice that, according to Lemma 7.3.4, the functions w_j, $j = 1,2$ are bounded in this case, (i.e., $C_1 = C_2 = 0$) while w_3 diverges at infinity with logarithmic growth. As a consequence, the resulting string solution no longer admits finite energy.

It is an interesting *open question* to know whether or not problem (7.2.4), (7.2.5), and (7.2.6) admits a solution when (7.2.18) is violated, or more precisely, when

$$\frac{2\lambda_3}{\lambda_2} < N + 1. \tag{7.3.43}$$

By our discussion it seems reasonable to expect an existence result to hold under the weaker assumption: $\frac{2\lambda_3}{\lambda_2} > N$. However, in this case, we see that the function w_3 admits a power growth at infinity, and so it fails to belong to Y_α. Therefore, a modified functional framework is required in order to handle this situation. On the other hand, the above discussion indicates that, as far as selfgravitating electroweak solutions are concerned, the condition $N + 1 < \frac{\sin^2\theta}{4\pi g\varphi_0^2}$ seems to be necessary when we wish to guarantee the finite energy properties (7.1.11) and (7.1.12).

7.4 Periodic electroweak vortices

In this section we focus our attention on solving (7.1.4) under the 'tHooft periodic boundary conditions, over the periodic cell domain

$$\Omega = \{z = \mathbf{a}_1 t + s\mathbf{a}_2 s,\ 0 < t, s < 1\},$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are two given linearly independent vectors. In other words, we search for a solution of (7.1.4) in Ω and subject to the boundary condition (2.1.38) on $\partial\Omega$.

As above, we will assign a set of N-points (counted with multiplicity) in Ω that correspond to the zeroes of the complex massive field W.

Thus, we call a *selfdual electroweak periodic N-vortex* any solution of (7.1.4) in Ω that satisfies (2.1.38) and for which W vanishes exactly at N-points (allowing multiplicity).

We shall see that the existence of selfdual electroweak periodic N-vortices imposes some (necessary) restrictions between the vortex number N and the physical parameters involved in the theory (see (7.4.1) below). Thus the effort is to construct periodic N-vortices within such constraints.

Unfortunately at the moment this has been possible only when $N = 1, 2, 3, 4$, while for $N \geq 5$ existence is ensured only under more restrictive conditions on the above mentioned parameters. More precisely we have:

Theorem 7.4.9 *([SY2],[BT2]) For a given $N \in \mathbb{N}$, we have:*
(i) A necessary condition for the existence of a selfdual electroweak periodic N-vortex solution of (7.1.4) and (2.1.38) in Ω, is that

$$g^2\varphi_0^2 < \frac{4\pi N}{|\Omega|} < \frac{g^2\varphi_0^2}{\cos^2\theta}. \tag{7.4.1}$$

(ii) Assume (7.4.1) and suppose in addition that

$$1 \neq \frac{4\pi N - g^2\varphi_0^2|\Omega|}{8\pi\sin^2\theta} < 2. \tag{7.4.2}$$

Then for a given set $\{z_1, \ldots, z_N\} \subset \Omega$ *(repeated according to their multiplicity), there exists a selfdual periodic* N*-vortex* (φ, W, P, Z) *satisfying* (7.1.4) *and* (2.1.38) *in* Ω *such that:* $\varphi > 0$ *and* W *vanishes exactly at* z_j *(according to its multiplicity)* $j = 1, \ldots, N$. *Furthermore, it admits a total flux* $\Phi = \frac{2\pi N}{e}$ *(see* (2.1.39)*), where* $-e = -(g \sin\theta)$ *is the electric charge.*

Theorem 7.4.9 was first established by Spruck–Yang in [SY2], under the more restrictive assumption: $\frac{4\pi N - g^2\varphi_0^2|\Omega|}{8\pi \sin^2\theta} < 1$.

In the form stated above, Theorem 7.4.9 is due to Bartolucci–Tarantello (cf. [BT2]).

Still when $\frac{4\pi N - g^2\varphi_0^2|\Omega|}{8\pi \sin^2\theta} = 1$. the question of solvability of (7.1.4) and (2.1.38) in Ω stands as a challenging *open problem*. As a matter of fact, each time we have:

$$\frac{4\pi N - g^2\varphi_0^2|\Omega|}{8\pi \sin^2\theta} \in \mathbb{N}, \tag{7.4.3}$$

we face additional analytical difficulties in the solvability of (7.1.4) and (2.1.38), for reasons that will become clear in the sequel. Indeed, we suspect that only the values in (7.4.3) represent a serious obstraction to the solvability of (7.1.4) and (2.1.38); and in fact, Theorem 7.4.9 should hold with (7.4.2) replaced by the condition:

$$\frac{4\pi N - g^2\varphi_0^2 N}{8\pi \sin^2\theta} \notin \mathbb{N}. \tag{7.4.4}$$

Proof of Theorem 7.4.9. Once more, we take advantage of the equivalent elliptic formulation corresponding to (7.1.4) and (2.1.38), as derived in Section 2.1 of Chapter 2. More precisely, defining new variables (u, v) such that

$$e^u = |W|^2 \qquad e^v = \varphi^2,$$

we need to solve:

$$\begin{cases} -\Delta u = 4g^2 e^u + g^2 e^v - 4\pi \displaystyle\sum_{j=1}^{N} \delta_{z_j} \text{ in } \Omega, \\ -\Delta v = \dfrac{g^2}{2\cos^2\theta}(\varphi_0^2 - e^v) - 2g^2 e^u \text{ in } \Omega, \\ u, v \text{ doubly periodic on } \partial\Omega, \end{cases} \tag{7.4.5}$$

in order to recover the whole vortex (W, φ, P, Z) solution for (7.1.4) and (2.1.38), by means of (2.1.22), (2.1.23), (2.1.24) and (1.4.18)–(1.4.19).

Concerning (7.4.5), observe that upon integration over Ω, every solution pair (u, v) must satisfy the contraints:

$$4g^2 \int_\Omega e^u + g^2 \int_\Omega e^v = 4\pi N, \qquad 2g^2 \int_\Omega e^u + \frac{g^2}{2\cos^2\theta} \int_\Omega e^v = \frac{g^2\varphi_0^2}{2\cos^2\theta}|\Omega|.$$

Consequently we derive,

$$\int_\Omega e^u = \frac{4\pi N - g^2\varphi_0^2|\Omega|}{4g^2\sin^2\theta}; \qquad \int_\Omega e^v = \frac{g^2\varphi_0^2|\Omega| - 4\pi N\cos^2\theta}{g^2\sin^2\theta}. \tag{7.4.6}$$

From (7.4.6) we immediately deduce (7.4.1) as a necessary condition for the solvability of (7.1.4) and (2.1.38), and part (i) is established. Next, we resume the function u_0 in (4.1.3), that together with u and v, we are going to consider as functions defined over the flat 2-torus $M = \mathbb{R}^2 / \mathbf{a}_1\mathbb{Z} \times \mathbf{a}_2\mathbb{Z}$. In this way, we decompose

$$\begin{aligned} u &= u_0 + w_1 + c : \int_M w_1 = 0 \text{ and } c = \fint_M u, \\ v &= w_2 + d : \int_M w_2 = 0 \text{ and } d = \fint_M v. \end{aligned} \tag{7.4.7}$$

From (7.4.6) it follows that:

$$\begin{aligned} e^c &= \left(\frac{4\pi N - g^2\varphi_0^2|\Omega|}{4g^2\sin^2\theta}\right)\frac{1}{\int_M e^{u_0+w_1}} \text{ and } e^d \\ &= \left(\frac{g^2\varphi_0^2|\Omega| - 4\pi N\cos^2\theta}{g^2\sin^2\theta}\right)\frac{1}{\int_M e^{w_2}}. \end{aligned} \tag{7.4.8}$$

Therefore, letting

$$\mu = 4\pi N - \frac{g^2\varphi_0^2|\Omega| - 4\pi N\cos^2\theta}{\sin^2\theta} = \frac{4\pi N - g^2\varphi_0^2|\Omega|}{\sin^2\theta}, \tag{7.4.9}$$

the necessary condition (7.4.1) then reads as

$$0 < \mu < 4\pi N. \tag{7.4.10}$$

Problem (7.4.5) can now be stated in terms of the new unknowns (w_1, w_2) equivalently as follows:

$$\begin{cases} -\Delta w_1 = \mu\dfrac{e^{u_0+w_1}}{\int_M e^{u_0+w_1}} + (4\pi N - \mu)\dfrac{e^{w_2}}{\int_M e^{w_2}} - \dfrac{4\pi N}{|M|} \text{ in } M \\ -\Delta w_2 = \dfrac{\mu}{2}\left(\dfrac{e^{u_0+w_1}}{\int_M e^{u_0+w_1}} - \dfrac{1}{|M|}\right) + \dfrac{4\pi N - \mu}{2\cos^2\theta}\left(\dfrac{e^{w_2}}{\int_M e^{w_2}} - \dfrac{1}{|M|}\right) \text{ in } M \\ w_1, w_2 \in H^1(M) : \displaystyle\int_M w_1 = 0 = \int_M w_2 \end{cases} \tag{7.4.11}$$

Clearly, problem (7.4.11) is a particular case of the general elliptic system (6.1.7), where $M = \mathbb{R}^2 / \mathbf{a}_1\mathbb{Z} \times \mathbf{a}_2\mathbb{Z}$, $\lambda = 4\pi N - \mu > 0$, $h = e^{u_0}$ and $f = 1$.

Therefore we can simply apply Corollary 6.1.3 to obtain a solution for (7.4.11), provided that

$$\mu \in \{(0, 8\pi) \cup (8\pi, 16\pi)\} \cap (0, 4\pi N). \tag{7.4.12}$$

Recalling the definition of μ in (7.4.9) from (7.4.12) we obtain the desired existence result as stated in (ii). □

Finally we notice that, for $N = 1, 2, 3, 4$, our existence result is rather "sharp" as we have:

Corollary 7.4.10 *If* $N = 1, 2$ *or* $N = 3, 4$ *and if* $\frac{4\pi N - g^2\varphi_0^2|\Omega|}{8\pi \sin^2\theta} \neq 1$, *then condition (7.4.1) is necessary and sufficient for the existence of a selfdual electroweak periodic* N*-vortex. Moreover, the* N*-points (counted with multiplicity) where* W *vanishes can be arbitrarily prescribed.*

Thus, we conclude our discussion about electroweak vortices with the following:
Open question: Does the conclusion of Corollary 7.4.10 also hold for $N \geq 5$, possibly by assuming (7.4.4)?

7.5 Concluding remarks

We end this monograph by drawing the reader's attention to other questions of interest in the study of gauge field vortices which were not touched upon by our analysis.

Firstly, we observe that all of the static selfdual vortex problems considered here could be more generally addressed on compact surfaces (cf. [KiKi]) other than the flat 2−torus, on which we have in fact focused since it is encountered more often in physical applications.

Actually it makes good sense to analyze the selfdual equations over compact surfaces with a boundary, where we assign Dirichlet boundary conditions of fixed "normal" state.

Secondly, concerning selfdual Chern–Simons vortices, it is important to clarify whether the selfdual solutions considered here in fact describe *all* finite energy solutions of the static Chern–Simons field equations: (1.2.33)–(1.2.34); (1.2.56); or (1.3.103)–(1.3.104) (subject to (1.3.116)–(1.3.117)). This would imply that Chern–Simons mixed vortex/antivortex configurations are not allowed, as indeed is true for the Maxwell–Higgs model. If not, see whether this is actually the case for a certain restricted class of solutions, such as local energy minimizers or "symmetric" solutions, as it happens for Yang–Mills fields.

Even more importantly, it would be helpful to accurately describe the nature of Chern–Simons vortices *away* from the selfdual regime, as was done for Ginzburg–Landau vortices (cf. [BBH], [DGP] and [PR]). For some attempt in this direction, see [HaK], [KS1] and [KS2].

Finally, our models are by nature bi-dimensional. To treat higher dimensional situations, it is necessary to consider other models with similar characteristics for example, the much acclaimed Seiberg–Witten model discussed in [DJLPW].

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